Notes on a Series of Lectures on Vector Calculus

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Defintion 1: Let $\mathbb{V}$ be a set. $\mathbb{V}$ is called a **vector space** provided that $\forall x, y, z \in \mathbb{V}$ and $\alpha, \beta \in \mathbb{R}$:

1. $x + y = y + x$
2. $\alpha(x + y) = \alpha x + \alpha y$
3. $(\alpha + \beta)x = \alpha x + \beta x$
4. $\exists 0 \in \mathbb{V} \ni x + 0$
5. $1 \cdot x = x$
6. $(x + y) + z = x + (y + z)$
7. $\exists \tilde{x} \ni x + \tilde{x} = 0$
8. $\alpha(\beta x) = (\alpha \cdot \beta)x$
9. $x + y \in \mathbb{V}$
10. $\alpha x \in \mathbb{V}$

Example 2: Put $\mathbb{V} := \{f : [0, 1] \to \mathbb{R} | \text{ f is continuous}\}$. For $f, g \in \mathbb{V}$, define addition as $(f + g)(x) := f(x) + g(x)$ and for $\alpha \in \mathbb{R}$, $f \in \mathbb{V}$, define multiplication as $(\alpha f)(x) := \alpha \cdot f(x)$. In this case, $\mathbb{V}$ is a vector space.

Example 3: Put $\mathbb{V} := \mathrm{M}_{2\times2}{}^{\mathbb{R}} = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right\}$

Example 4: Put $\mathbb{V} := \mathbb{R}^n$ where $\mathbb{R}^n := \{(x_1, x_2, ..., x_n) : x_i \in \mathbb{R}, 1 \leq i \leq n\}$. Then define addition as $(x_1, ..., x_n) + (y_1, ..., y_n) := (x_1 + y_1, ..., x_n + y_n)$ and define multiplication as $\alpha(x_1, ..., x_n) := ((\alpha x_1, ..., \alpha x_n))$.

Definition 5: Let $\mathbb{V}$ be a vector space and let $\mathbb{W} \subseteq \mathbb{V}$. If $\mathbb{W}$ is itself is a vector space, then $\mathbb{W}$ is called a **subspace**.

Observation 6: To show that $\mathbb{W}$, as above, is a subspace, it suffices to show that:

1. $0 \in \mathbb{W}$
2. $x + y \in \mathbb{W}$ wherever $x, y \in \mathbb{W}$
3. $\alpha x \in \mathbb{W}$ wherever $x \in \mathbb{W}, \alpha \in \mathbb{R}$

Example 7: Indentify the subspaces of $\mathbb{R}^2$.

1. $\{\vec{0}\} = \{(0, 0)\}$
2. $\mathbb{R}^2$

3. $\mathcal{F} := \{y = kx : k \text{ is a fixed real number}\}, x \in \mathbb{R}$

Definition 8: Let $\mathbb{V}$ be a vector space. Let $\mathrm{A} = v_1, ..., v_n \subseteq \mathbb{V}$. Then $\mathrm{span}(\mathrm{A}) := \{\sum_{i=1}^{n} a_i v_i : a_i \in \mathbb{R}, v_i \in \mathrm{A}\}$.

Example 9: Consider $\mathbb{V} = \mathbb{R}^2$. In this case, $\mathrm{span}(1,1) = (x, y) : x = y$

Proposition 10: If $\mathbb{V}$ is a vector space and $\mathrm{A} \subseteq \mathbb{V}$, then span(A) is a subspace of $\mathbb{V}$.

Example 11:

$$\begin{aligned} \vec{v}_1 =& (1,1) \\ \vec{v}_2 =& (-1,-1) \\ \vec{v}_3 =& (1,2) \end{aligned}$$

Note that $\vec{v}_1$ and $\vec{v}_2$ are opposites. That is, $\vec{v}_1 = -\vec{v}_2 \iff \vec{v}_1 + \vec{v}_2 = 0$. Is there a $k \in \mathbb{R}$ such that $\vec{v}_1 = k\vec{v}_3$? No there isn't. Note that $\vec{v}_1$ and $\vec{v}_2$ are parallel while $\vec{v}_1$ and $\vec{v}_3$ are not parallel. Also, the equation $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ has a nontrivial solution in the complex integers. Finally, the equation $c_1\vec{v}_1 + c_3\vec{v}_3 = \vec{0}$ has only the trivial solution in the complex integers (i.e. make everything 0).

Definition 12: Let $\mathbb{V}$ be a vector space. Let $\mathrm{A} = v_1, ..., v_n \subseteq \mathbb{V}$. Then A is a linearly independent set if and only if the only solution to $\sum_{i=1}^{n} c_i v_i = \vec{0}$ is the trivial one. Otherwise, A is linearly dependent.

Example 13: Consider $\vec{v} = (a, b)$. $\vec{v} = a(1,0) + b(0,1)$ and this is the unique solution given you have to use these vectors and pick their scalar coefficients. These are orthonormal vectors as they are orthogonal and they are both of unit length. Note that these vectors are linearly independent. It takes exaclty 2 linearly independent vectors like this to define any vector in $\mathbb{R}^2$. For example, we can get the vector $\vec{v} = (1,2)$ by doing this: $\vec{v} = 1(1,1) + 1(0,1)$. Basically, $\gamma = \vec{e}_1, \vec{e}_2$ where the two vectors are linearly independent generates $\mathbb{R}^2$. Alternatively, you could say that $\mathrm{span}\vec{e}_1, \vec{e}_2 = \mathbb{R}^2$.

Definition 14: Given a vector space $\mathbb{V}$ if $\alpha \subseteq \mathbb{V}$ is a linearly independent, generating set, then α is called a **basis** of $\mathbb{V}$.

Definition 15: Let $\mathbb{V}$ be a vector space. If β is a basis for $\mathbb{V}$, then $\dim \mathbb{V} := |\beta| = \mathrm{card}(\beta)$.

Theorem 16: $\dim \mathbb{R}^n = n$.

Example 17:

$$\begin{aligned} \vec{a} =& (1,2,3) \\ \vec{b} =& (1,3,4) \end{aligned}$$

In this case $\vec{a} + \vec{b} = (2,5,7)$ and $\vec{a} - 2\vec{b} = (-1,-4,-5)$. Also, $||\vec{a}||_2 = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.

Example 18:

$$\begin{aligned} \vec{a} =& (1,2) \\ \vec{b} =& (3,-2) \end{aligned}$$

In this case $\vec{a}+2\vec{b}=(1,2)+(6,-4)=(7,-2)$ and $||\vec{a}||_2=\sqrt{1^2+2^2}=\sqrt{5}$.
Defintion 19: Let $\vec{a},\vec{b}\in\mathbb{R}^n$, say $\vec{a}=(a_1,...,a_n)$, $\vec{b}=(b_1,...,b_n)$. Then the **dot** or **scalar** or **inner product** is denoted $\vec{a}\cdot\vec{b}$, or $<\vec{a},\vec{b}>$ and is defined to be $\sum_{j=1}^{n} a_j b_j$. For example $\vec{a}\cdot\vec{b}=(1,2)\cdot(3,-2)=1\cdot 3+2\cdot(-2)=-1$ using the vectors as above. Note that $<\vec{a},\vec{a}>=\vec{a}\cdot\vec{a}\geq 0$ and $<\vec{a},\vec{a}>=0 \iff \vec{a}=\vec{0}$.
Theorem 20: Let $\vec{a},\vec{b}\in\mathbb{R}^n$. Then:

1. $\vec{a}\cdot\vec{a}=0 \iff \vec{a}=0$ and also $\vec{a}\cdot\vec{a}\geq 0,\forall\vec{a}$
2. $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$
3. $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$
4. $(k\vec{a})\cdot\vec{b}=k(\vec{a}\cdot\vec{b})=\vec{a}\cdot(k\vec{b}),\forall k\in\mathbb{R}$

Example 21: $(\vec{e}_1+2\vec{e}_2+...+n\vec{e}_n)\cdot(\vec{e}_1-\vec{e}_2+\vec{e}_3-\vec{e}_4+...+(-1)^{n+1}\vec{e}_n)=(1,2,3,...,n)\cdot(1,-1,1,...,(-1)^{n+1})=1-2+3-4+...+(-1)^{n+1}n$
Definition 22: Let $\vec{a}\in\mathbb{R}^n$. Then the **2-norm** of $\vec{a}$ is $||\vec{a}||_2:=\sqrt{{a_1}^2+{a_2}^2+...+{a_n}^2}$.
Theorem 23: For any $\vec{a}\in\mathbb{R}^n$, ${||\vec{a}||_2}^2=\vec{a}\cdot\vec{a}$.
Theorem 24: If $\vec{a}$ and $\vec{b}$ are any two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, $\vec{a}\cdot\vec{b}=||\vec{a}||||\vec{b}||\cos\theta$.
Proof: Put $\vec{c}:=\vec{b}-\vec{a}$. Note that if either $\vec{a}$ or $\vec{b}$ is $\vec{0}$, then the claim is trivially true. Suppose not. By the law of cosines

$$\begin{aligned}
||\vec{c}||^2 =&||\vec{a}||^2+||\vec{b}||^2-2||\vec{a}||||\vec{b}||\cos\theta\\
2||\vec{a}||||\vec{b}||\cos\theta =&||\vec{a}||^2+||\vec{b}||^2-||\vec{c}||^2\\
2||\vec{a}||||\vec{b}||\cos\theta =&\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}-(\vec{b}-\vec{a})\cdot(\vec{b}-\vec{a})\\
2||\vec{a}||||\vec{b}||\cos\theta =&\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}-\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{a}\\
2||\vec{a}||||\vec{b}||\cos\theta =&2\vec{a}\cdot\vec{b}\\
||\vec{a}||||\vec{b}||\cos\theta =&\vec{a}\cdot\vec{b}
\end{aligned}$$

Corollary 25: The angle θ between two vectors $\vec{a},\vec{b}\in\mathbb{R}^2$ or $\mathbb{R}^3$ is given by $\theta=\arccos\left(\frac{\vec{a}\cdot\vec{b}}{||\vec{a}||||\vec{b}||}\right)$, provided $\vec{a},\vec{b}\neq 0$.
Corollary 26: Given $\vec{a},\vec{b}\in\mathbb{R}^2$ or $\mathbb{R}^3$, $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a}\cdot\vec{b}=0$.
Theorem 27 (Cauchy-Schwarz): For all vectors $\vec{a},\vec{b}\in\mathbb{R}^n$, $|\vec{a}\cdot\vec{b}|\leq||\vec{a}||\cdot||\vec{b}||$.
Example 28:

$$\begin{aligned}
\vec{a} =&(1,2,3)\\
\vec{b} =&(-1,0,1)
\end{aligned}$$

$$\vec{a} \cdot \vec{b} = 2$$
$$||\vec{a}|| = \sqrt{14}$$
$$||\vec{b}|| = \sqrt{2}$$

$$|\vec{a} \cdot \vec{b}| = 2 \leq \sqrt{28} = ||\vec{a}|| \cdot ||\vec{b}||$$

Proposition 29: Hey look this is cool.

1. $\vec{i} \times \vec{j} = \vec{k}$

2. $\vec{j} \times \vec{k} = \vec{i}$

3. $\vec{k} \times \vec{i} = \vec{j}$

Definition 30: Let $\vec{a}, \vec{b} \in \mathbb{R}^3$. Then the **cross product** of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$ is the vector whoes length and direction are:

1. $||\vec{a} \times \vec{b}|| := ||\vec{a}|| \cdot ||\vec{b}|| \sin \theta$

2. $\vec{a} \times \vec{b}$ is mutually orthogonal to $\vec{a}$ and $\vec{b}$ and the triple $(\vec{a}, \vec{b}, \vec{a} \times \vec{b})$ forms a right handed system.

Lemma 31: Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ and $k \in \mathbb{R}$ be any scalar. Then

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

2. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

3. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

4. $k(\vec{a} \times \vec{b}) = (k\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$

Put

$$\vec{a} := a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

and

$$\vec{b} := b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$$

Then

$$\begin{aligned}\vec{a}\times\vec{b} =&(a_1\vec{i}+a_2\vec{j}+a_3\vec{k})\times(b_1\vec{i}+b_2\vec{j}+b_3\vec{k})\\ =&a_1b_1\vec{i}\times\vec{i}+a_1b_2\vec{i}\times\vec{j}+a_1b_2\vec{i}\times\vec{k}+\\ &+a_2b_1\vec{j}\times\vec{i}+a_2b_2\vec{j}\times\vec{j}+a_1b_3\vec{j}\times\vec{k}+\\ &+a_3b_1\vec{k}\times\vec{i}+a_3b_2\vec{k}\times\vec{j}+a_3b_3\vec{k}\times\vec{k}\\ =&0+a_1b_2\vec{k}-a_1b_2\vec{j}+\\ &-a_2b_1\vec{k}+0+a_1b_3\vec{i}+\\ &+a_3b_1\vec{j}-a_3b_2\vec{i}+0\\ =&(a_2b_3)\vec{i}+(a_3b_1-a_1b_3)\vec{j}+(a_1b_2-a_2b_1)\vec{k}\\ =&\begin{vmatrix} a_2 & a_3\\ b_2 & b_3\end{vmatrix}\vec{i}-\begin{vmatrix} a_1 & a_3\\ b_1 & b_3\end{vmatrix}\vec{j}+\begin{vmatrix} a_1 & a_2\\ b_1 & b_2\end{vmatrix}\vec{k}\\ =&\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_2 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\end{aligned}$$

Proposition 32:

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_2 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Example 33: *(an example for Proposition 32)*

Definition 34: Given $\vec{a},\vec{b},\vec{c}\in\mathbb{R}^3$, the volume of a paralohyphid is equal to the area of the base times the height, which can be computed using the **triple scalar product**.

$$\begin{aligned}V =&||\vec{a}\times\vec{b}||||\vec{c}||\,|\cos\theta|\\ =&|(\vec{a}\times\vec{b})\cdot\vec{c}|\\ =&\left|\begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & b_3\end{vmatrix}\right|\end{aligned}$$

Proposition 35: The vector parametric equation for the line containing $P_0(b_1,b_2,b_3)$, whose position vector is $O\vec{P}_0=\vec{b}=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$, and parallel to $\vec{a}=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ is

$$\vec{r}(t)=\vec{b}+t\vec{a}, t\in\mathbb{R}$$

that is,

$$\vec{r}(t) = \left\{ \begin{array}{l} x(t) = b_1 + ta_1 \\ y(t) = b_2 + ta_2 \\ z(t) = b_3 + ta_3 \end{array} \right\}$$

Example 36: *(an example for Proposition 35)*
Platform $9\frac{3}{4}$**:** Given vectors $\vec{P_0}(x_0, y_0, z_0)$ and $\vec{P}(x, y, z)$, $\vec{P_0P} = (x - x_0, y - y_0, z - z0)$ and a normal vector $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$, we can define the plane of these two vectors as

$$0 = \vec{n} \cdot \vec{P_0P} = A(x - x_0) + B(y - y_0) + C(z - z_0)$$

or

$$Ax + By + Cz = D$$

where

$$D := Ax_0 + By_0 + Cz_0$$

Example 37: A plane contains $(3, -1, 2)$ and is normal to (1,-1,2). Then the plane is

$$1(x - 3) - 1(y + 1) + 2(z - 2) = 0$$

or

$$z - y + 2z = 8$$

Example 38: A plane contains the points $P = (3, -1, 2)$, $Q = (2, 0, 5)$, and $R = (1, -2, 4)$. Then

$$\begin{aligned} \vec{PQ} &= (-1, 1, 3) \\ \vec{RQ} &= (1, 2, 1) \end{aligned}$$

$$\begin{aligned} \vec{PQ} \times \vec{RQ} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 3 \\ 1 & 2 & 1 \end{vmatrix} \\ &= -5\vec{i} + 4\vec{j} - 3\vec{k} \end{aligned}$$

$$\therefore 0 = -5(x - 3) + 4(y + 1) - 3(z - 2)$$

or

$$-5x + 4y - 3z = -25$$

Return to Platform $9\frac{3}{4}$:
Given $\vec{c}$ (a point in the plane) in addition to $\vec{a}$ and $\vec{b}$ in the plane not parallel, then we get a plane

$$\vec{x}(s,t) = \vec{c} + t\vec{a} + 3\vec{b}, \forall s,t \in \mathbb{R}$$

$$\vec{x}(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix} = \begin{bmatrix} c_1 + ta_1 + sb_1 \\ c_2 + ta_2 + sb_2 \\ c_3 + ta_3 + sb_3 \end{bmatrix}$$

Projections are also cool.

$$\begin{aligned} |\cos\theta| =& \frac{||\text{proj}_{\vec{b}}\vec{a}||}{||\vec{a}||} \\ ||\text{proj}_{\vec{b}}\vec{a}|| =& ||\vec{a}|| \cdot |\cos\theta| \\ ||\text{proj}_{\vec{b}}\vec{a}|| =& \frac{||\vec{a}|| \cdot ||\vec{b}|| \cdot |\cos\theta|}{||\vec{b}||} \\ ||\text{proj}_{\vec{b}}\vec{a}|| =& \frac{|\vec{a} \cdot \vec{b}|}{||\vec{b}||} \end{aligned}$$

$$\therefore \text{proj}_{\vec{b}}\vec{a} = \frac{\vec{b}}{||\vec{b}||} \cdot \frac{\vec{a} \cdot \vec{b}}{||\vec{b}||} = \left(\frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \cdot \vec{b} \right)$$

Theorem 39: Let $\vec{a}, \vec{b} \in \mathbb{R}^3$, then the projection of $\vec{a}$ onto $\vec{b}$, denoted $\text{proj}_{\vec{b}}\vec{a}$, is given by

$$\text{proj}_{\vec{b}}\vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \cdot \vec{b} \right)$$

Platform $9\frac{3}{4}$ Again:
Some polar review:

$$\text{polar to rectangular} \begin{Bmatrix} x &=& r\cos\theta \\ y &=& r\sin\theta \end{Bmatrix}$$
$$\text{rectangular to polar} \begin{Bmatrix} r^2 &=& x^2 + y^2 \\ \theta &=& \arctan\left(\frac{y}{x}\right) \end{Bmatrix}$$

Now cylidrical:

$$\text{cylidrical to rectangular}\begin{Bmatrix} x & = & r\cos\theta \\ y & = & r\sin\theta \\ z & = & z \end{Bmatrix}$$

$$\text{rectangular to cylindrical}\begin{Bmatrix} r^2 & = & x^2+y^2 \\ \theta & = & \arctan\left(\frac{y}{x}\right) \\ z & = & z \end{Bmatrix}$$

Spherical is hard mode:

$$\text{spherical to rectangular}\begin{Bmatrix} x & = & \rho\sin\phi\cos\theta \\ y & = & \rho\sin\phi\sin\theta \\ z & = & \rho\cos\phi \end{Bmatrix}$$

$$\text{rectangular to spherical}\begin{Bmatrix} \rho^2 & = & x^2+y^2+z^2 \\ \tan\phi & = & \frac{\sqrt{x^2+y^2}}{z} \\ \tan\theta & = & \frac{y}{x} \end{Bmatrix}$$

$$\text{spherical to cylindrical}\begin{Bmatrix} r & = & \rho\sin\phi \\ \theta & = & \theta \\ z & = & \rho\cos\phi \end{Bmatrix}$$

$$\text{cylindrical to spherical}\begin{Bmatrix} \rho^2 & = & r^2+z^2 \\ \tan\phi & = & \frac{r}{z} \\ \theta & = & \theta \end{Bmatrix}$$

Definition 40: Let $\vec{x} \in \mathbb{R}^n$ and let $\epsilon > 0$. Then an **ϵ-neighborhood** or the **open ball of radius ϵ centered at $\vec{x}$** is

$$\mathcal{B}_\epsilon(\vec{x}) := \{\vec{y} \in \mathbb{R}^n : ||\vec{x} - \vec{y}|| < \epsilon\}$$

Moreover, the deleted ϵ-neighborhood of $\vec{x}$ is denoted variously by

$$\mathcal{B}_\epsilon(\vec{x})\backslash\{\vec{x}\} = \mathcal{B}^*_\epsilon(\vec{x}) := \{\vec{y} \in \mathbb{R}^n : 0 < ||\vec{x} - \vec{y}|| < \epsilon\}$$

Definition 41: Let $S \subseteq \mathbb{R}^n$. A point $\vec{x}_0 \in S$ is an **interior point** of S if $\forall\epsilon > 0, B_\epsilon(\vec{x}_0) \subseteq S$. A point $\vec{x}_0 \in S$ is a **boundary point** of S if $\forall\epsilon > 0, S \cap B_\epsilon(\vec{x}_0) \neq \emptyset$ and $(\mathbb{R}^n\backslash S) \cap B_\epsilon(\vec{x}_0) \neq \emptyset$. The boundary of a set is denoted variously as bd $S = \partial S$.

Definition 42: Let $S \subseteq \mathbb{R}^n$. Then:

1. S is open if bd $S \subseteq \mathbb{R}^n\backslash S$

2. S is closed if bd $S \subseteq S$

Theorem 43: Let $S \subseteq \mathbb{R}^n$

1. S is open $\iff$ $S = \text{int } S$

2. S is closed $\iff$ $\mathbb{R}^n \backslash S$ is open

Theorem 44: Let $\{C_n\}_{n\in\mathcal{A}}$ be an arbitrary collection of closed sets and let $\{O_n\}_{n\in\mathcal{B}}$ be an arbitrary collection of open sets. Then

1. $\cup_{n\in\mathcal{B}_0\subseteq\mathcal{B}} O_n$ is open

2. $\cap_{n\in\mathcal{A}_0\subseteq\mathcal{A}} C_n$ is closed

Example 45: *(an example for Theorem 44)*
Definition 46: Let $S \subseteq \mathbb{R}^n$. We say that $\vec{x}_0 \in \mathbb{R}^n$ is a **limit point** of S provided that $\forall \epsilon > 0$, $B_\epsilon(\vec{x}_0) \cap S$ has infinitely many points in it. The closure of S is defined to be $S \cup \{\vec{x} \in \mathbb{R}^n : \vec{x}$ is a linit point of $S\}$. We denote the closure of S by $\bar{S}$.
Example 47: $S = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}$ Is S open? Is it closed? Note that int $S = \{\emptyset\}$, bd $S = S \cup \{0\}$, and $\bar{S} = S \cup \{0\}$. Thus S is not open and S is not closed. Now $S = \{\vec{x} \in \mathbb{R}^3 : ||\vec{x}||^2 < 9\}$. S is open. S is not closed. Furthermore, bd $S = \{\vec{x} \in \mathbb{R}^3 : ||\vec{x}||^2 = 9\}$, int $S = S$, and the set of all limit points of S is $S \cup \text{bd } S$.
Definition 48: Let $S \subseteq \mathbb{R}^n$. Then S is **compact** provided that given any open cover $\mathcal{F}$ of S, there is a finite subcollection $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $\mathcal{F}_0$ is also an open cover of S. $\mathcal{F}_0$ is called a finite subcover.
Example 49: ITT we prove that this set is not compact

$$S = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

$$\mathcal{F} = \left\{\left(\frac{1}{n}, 2\right)\right\}_{n=1}^{\infty}$$

$$\cup_{\mathcal{F}} = (0, 2) \supseteq S$$

$$\cup_{n=1}^{k}\left(\frac{1}{n}, 2\right) = \left(\frac{1}{k}, 2\right) \not\supseteq S$$

Theorem 50 (Heine-Borel): If $S \subseteq \mathbb{R}^n$, then S is compact if and only if S is both closed and bounded.
Theorem 51 (Bolzano-Weierstrass): If a bounded set $S \subseteq \mathbb{R}^n$ contains infinitely many points then there exists at least one point in $\mathbb{R}^n$ that is an **accumulation point** of S. $\vec{x}$ is an **accumulation point** of S if every deleted neighborhood of $\vec{x}$ contains a point of S.

Proof: Let S be a bounded subset of $\mathbb{R}^n$ containinng infinitely many points. Suppose, for contradiction, that S has no accumulation points. Then S is closed. (Why? Because a closed set, by theorem, contains all of its accumulation points.) By Heine-Borel, S is compact. Observe that as each $\vec{x} \in S$ is not an accumulation point, there is some neighborhood $\mathcal{B}_\epsilon(\vec{x}) \ni \mathcal{B}_\epsilon(\vec{x}) \cap S = \{\vec{x}\}$. Note that $\mathcal{F} = \{\mathcal{B}_\epsilon(\vec{x})\}$ is an open cover for S. But then we can extract a finite subcover because S is compact – say, $\mathcal{F}_0 = \{\mathcal{B}_{\epsilon_i}(\vec{x}_i)\}_{i=1}^n \ni \cup_{i=1}^n \mathcal{B}_{\epsilon_i}(\vec{x}_i) \supseteq S$. But $(\cup_{i=1}^n \mathcal{B}_{\epsilon_i}(\vec{x}_i)) \cap S = \{\vec{x}_1, \vec{x}_2, ..., \vec{x}_n\} \neq S$ but it was supposed to be by definition of a compact set.

Definition 52: The range of a function $f : \mathbb{X} \to \mathbb{Y}$ is the set

$$\text{rng } f := \{y \in \mathbb{Y} : \exists x \in \mathbb{X} \ni f(x) = y\}$$

Defention 53: A function $f : \mathbb{X} \to \mathbb{Y}$ is said to be **onto** or **surjective** if rng $f = \mathbb{Y}$. f is said to be **one-to-one** or **injective** if given $y \in \text{rng } f, \exists! x \in \mathbb{X} \ni f(x) = y$ or, equivelantly, whenever $f(x_1) = f(x_2)$, $x_1, x_2 \in \mathbb{X}$, $x_1 = x_2$. Finally, f is **bijective** is it is one-to-one, onto.

Example 54: A scalar-valued functon.

$$T(x, y, z) = \frac{1}{\sqrt{4 - x^2 - y^2 - z^2}}$$

$$\begin{aligned} \text{dom } f &= \mathcal{B}_2(\vec{0})\{\vec{x} \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 4\} \\ &= \mathcal{B}_2 \\ \text{rng } f &= \left[\frac{1}{2}, \infty\right) \end{aligned}$$

$$T : \mathcal{B}_2(\vec{0}) \to \mathbb{R}$$

Example 55: A vector-valued function

$$\vec{T}(x, y) = (x + y, x + 2y, y)$$

$$\vec{T} : \mathbb{R}^2 \to \mathbb{R}^3$$

$$\begin{aligned} \text{dom } \vec{T} =& \mathbb{R}^2 \\ \text{rng } \vec{T} =& ??? \end{aligned}$$

Example 56: Another scalar-cvalued function that results in a paraboloid.

$$f(x, y) = z = x^2 + y^2$$

$$f : \mathbb{R}^2 \to \mathbb{R}$$

What's Quadric Surfaces, Precious?

$$Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 + Gx + Hy + Jz + K = 0$$

Where A

Example 57: An ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Definition 58: Let $\vec{f} : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Then $\lim_{x \to \vec{a}} \vec{f}(\vec{x}) = \vec{L} \iff \forall \epsilon > 0, \exists \delta > 0 \ni ||\vec{f}(\vec{x}) - \vec{L}|| < \epsilon$ whenever $||\vec{x} - \vec{a}|| < \delta$, where $\vec{x} \in \mathbb{X}$.

Remark In Definition 58, $\vec{a} \in \bar{\mathbb{X}}$.

Theorem 59 (Uniqueness of Limits): Let $f : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$. If $\lim_{x \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}$ and $\lim_{x \to \vec{a}} \vec{f}(\vec{x}) = \vec{M}$ then $\vec{L} = \vec{M}$.

Proof: Let $\epsilon > 0$ be given. Because $\lim_{x \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}$, by Definition 58, it follows that $\exists \delta_1 > 0 \ni ||\vec{f}(\vec{x}) - \vec{L}|| < \epsilon$ whenever $||\vec{x} - \vec{a}|| < \delta_1$. Similarly, $\exists \delta_2 > 0 \ni ||\vec{f}(\vec{x}) - \vec{M}|| < \epsilon$ whenever $||\vec{x} - \vec{a}|| < \delta_2$. Put $\delta_0 := \min\{\delta_1, \delta_2\}$. Then if $||\vec{x} - \vec{a}|| < \delta_0$,

$$\begin{aligned} ||\vec{L} - \vec{M}|| &= ||\vec{f}(\vec{x}) - vecM - \vec{f}(\vec{x}) + \vec{L}|| \\ &\leq ||\vec{f}(\vec{x}) - \vec{L}|| + ||\vec{f}(\vec{x}) - \vec{M}|| \\ &< 2\epsilon \end{aligned}$$

So, by the fact that $\epsilon > 0$ is arbitrary, it follows that $||\vec{L} - \vec{M}|| = 0$, whence by the positive definiteness of the norm, we get that $\vec{L} = \vec{M}$.

Theorem 60: Let $\vec{F}, \vec{G} : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $f, g : \mathbb{X} \in \mathbb{R}^n \to \mathbb{R}$. Let $k \in \mathbb{R}$.

1. If $\lim_{x \to \vec{a}} \vec{F}(\vec{x}) = \vec{L}$ and $\lim_{x \to \vec{a}} \vec{G}(\vec{x}) = \vec{M}$, then $\lim_{x \to \vec{a}} (\vec{F}(\vec{x}) \pm \vec{G}(\vec{x})) = \vec{L} \pm \vec{M}$.
2. If $\lim_{x \to \vec{a}} f(\vec{x}) = L$ and $\lim_{x \to \vec{a}} g(\vec{x}) = M$, then $\lim_{x \to \vec{a}} (f \pm g)(\vec{x}) = L \pm M$.
3. If $\lim_{x \to \vec{a}} \vec{F}(\vec{x}) = \vec{L}$, then $\lim_{x \to \vec{a}} k\vec{F}(\vec{x}) = k\vec{L}$.
4. If $\lim_{x \to \vec{a}} f(\vec{x}) = L$, then $\lim_{x \to \vec{a}} kf(\vec{x}) = kL$.
5. If $\lim_{x \to \vec{a}} f(\vec{x}) = L$ and $\lim_{x \to \vec{a}} g(\vec{x}) = M$, then
   (a) $\lim_{x \to \vec{a}} fg(\vec{x}) = LM$
   (b) $\lim_{x \to \vec{a}} \frac{f}{g}(\vec{x}) = \frac{L}{M}, M \neq 0$

Proof:

1. Let $\epsilon > 0$ be given. As $\lim_{\vec{x}\to\vec{a}} \vec{F}(\vec{x}) = \vec{L}$ and $\lim_{\vec{x}\to\vec{a}} \vec{G}(\vec{x}) = \vec{M}$, there exist $\delta_1, \delta_2 > 0 \ni ||\vec{F}(\vec{x}) - \vec{L}|| < \frac{\epsilon}{2}$ whenever $||\vec{x} - \vec{a}|| < \delta_1$ and $||\vec{G}(\vec{x}) - \vec{M}|| < \frac{\epsilon}{2}$ whenever $||\vec{x} - \vec{a}|| < \delta_2$. Put $\delta_0 = \min\{\delta_1, \delta_2\}$. Then whenever $||\vec{x} - \vec{a}|| < \delta_0$,

$$\begin{aligned} ||(\vec{F}(\vec{x}) + \vec{G}(\vec{x})) - (\vec{L} - \vec{M})|| &\leq ||\vec{F}(\vec{x}) - \vec{L}|| + ||\vec{G}(\vec{x}) - \vec{M}|| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

5.i. Let $\epsilon < 0$ be given. As $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ and $\lim_{\vec{x}\to\vec{a}} g(\vec{x}) = M$, it follows that $\exists \delta_1, \delta_2 > 0 \ni$ whenever $||\vec{x} - \vec{a}|| < \delta_1, |f(\vec{x}) - L| < \frac{\epsilon}{2|L|+1}$ and $||\vec{x} - \vec{a}|| < \delta_2, |g(\vec{x}) - M| < \frac{\epsilon}{2\alpha_0+1}$. Note that

$$\begin{aligned} |f(\vec{x})g(\vec{x}) - LM| &= |f(\vec{x})g(\vec{x}) - Lg(\vec{x}) + Lg(\vec{x}) - LM| \\ &\leq |(f(\vec{x}) - L)g(\vec{x})| + |(g(\vec{x}) - M)L| \\ &= |g(\vec{x})||f(\vec{x}) - L| + |L||g(\vec{x}) - M| \end{aligned}$$

Also note that as $|g(\vec{x}) - M| < \frac{\epsilon}{2}$ for $||\vec{x} - \vec{a}|| < \delta_2$, it follows that $|g(\vec{x})| \leq |g(\vec{x}) - M| + |M| < \frac{\epsilon}{2\alpha_0+1} + |M| =: \alpha_0$, whenever $||\vec{x} - \vec{a}|| < \delta_2$. Finally, put $\delta_0 = \min\{\delta_1, \delta_2\}$. Then whenever $||\vec{x} - \vec{a}|| < \delta_0$, $|(fg)(\vec{x}) - LM| \leq |g(\vec{x})||f(\vec{x}) - L| + |L||g(\vec{x}) - M| < \alpha_0 \frac{\epsilon}{2\alpha_0+1} + |L|\frac{\epsilon}{2|L|+1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. **NB: THERE IS SOMETHING WRONG WITH THIS PROOF according to Jeffmin.**

Example 61:

1. $\lim_{(x,y,z)\to(0,0,0)}(x^2 + 2xy + yz + z^3 + 2) = 2$

2. $\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y} = \lim_{(x,y)\to(0,0)}(x + y) = 0$

Example 62:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

$x = 0:\ \lim_{y\to 0} \frac{-y^2}{y^2} = -1$

$y = 0:\ \lim_{x\to 0} \frac{x^2}{x^2} = 1$

Since these are not equal the limit does not exist.

Example 63:

$$\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2 + y^4)^3}$$

$$\begin{aligned}
\lim_{(x,mx)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3} &= \lim_{x\to 0} \frac{m^4x^8}{(x^2+m^4x^4)^3} \\
&= \lim_{x\to 0} \frac{m^4x^8}{(x^2)^3(1+m^4x^2)^3} \\
&= \lim_{x\to 0} \frac{m^4x^2}{(1+m^4x^2)^3} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\lim_{(y^2,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3} &= \lim_{y\to 0} \frac{y^{12}}{(y^4+y^4)^3} \\
&= \lim_{y\to 0} \frac{y^{12}}{8y^{12}} \\
&= \frac{1}{8}
\end{aligned}$$

Since these are not equal the limit does not exist.

Theorem 64: Suppose that $\vec{f} : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a vector-valued function. Then

$$\begin{aligned}
\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) &= \lim_{\vec{x}\to\vec{a}} (f_1(\vec{x}), ..., f_n(\vec{x})) \\
&= (\lim_{\vec{x}\to\vec{a}} f_1(\vec{x}), ..., \lim_{\vec{x}\to\vec{a}} f_n(\vec{x})) \\
&= (x_1, ..., x_n) \\
&:= \vec{L}
\end{aligned}$$

assuming $\lim_{f_i}(\vec{x}) = x_i$ for $1 \leq i \leq m$.

Example 65: *(A cool example involving matrices for Theorem 64.)*

Remark: $f(c_1\vec{x} + c_2\vec{y}) = c_1 f(\vec{x}) + c_2 f(\vec{y}) \forall c_1, c_2 \in \mathbb{R}, \forall \vec{x}, \vec{y} \in \mathbb{R}^n$

Definition 66 (Continuity in $\mathbb{R}^n$): Let $f : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$. If $\vec{a} \in \mathbb{X}$, then f is **continuous** at $\vec{a}$ provided that $\forall \epsilon > 0, \exists \delta > 0 \ni ||f(\vec{x}) - f(\vec{a})|| < \epsilon$ whenever $||\vec{x} - \vec{a}|| < \delta$.

Proposition 67: Suppose $\vec{F} : \vec{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$, U open, and $\vec{x}_0 \in \vec{U}$. $\vec{F}$ is continuous at $\vec{x}_0 \iff \lim_{\vec{x}\to\vec{x}_0} \vec{F}(\vec{x}) = \vec{F}(\vec{x}_0)$.

Example 68: *(A cool example involving Example 65 for continuity.)*

Example 69:

$$f(x, y, z) = x^3 + 3xy^2 + yz^3 + 2$$

$$f : \mathbb{R}^3 \to \mathbb{R}$$

Therefore f is continuous on $\mathbb{R}^3$ because

$$\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0)$$

Example 70:

$$f(x,y) := \left\{ \begin{array}{ll} \frac{x^2-y^2}{x^2+y^2} & ,\vec{x} \neq \vec{0} \\ 0 & ,\vec{x} = \vec{0} \end{array} \right\}$$

$$\lim_{(x,mx)\to(0,0)} \frac{x^2 - m^2x^2}{x^2 + m^2x^2} = \frac{1-m^2}{1+m^2}$$

Therefore the limit does not exist because it depends on m. Therefore f is not continuous at $\vec{x} = 0$. But f is continuous on $\mathbb{R}^2 \backslash \{\vec{0}\}$.

Example 71:

$$\begin{aligned} \lim_{\vec{x}\to\vec{0}} \frac{x^2}{x^2+y^2} &= \lim_{r\to 0} \frac{r^2\cos^2\theta}{r^2} \\ &= \lim_{r\to 0} \cos^2\theta \\ &= \cos^2\theta \end{aligned}$$

Therefore the limit does not exist because it depends on θ.

Theorem 72: If $\vec{f} : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\vec{g} : \mathbb{Y} \subseteq \mathbb{R}^m \to \mathbb{R}^p$ are continuous on rng $\vec{f} \subseteq \mathbb{Y}$, then $\vec{g} \circ \vec{f} : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^p$ is defined and continuous.

Proof: Given $\epsilon > 0$, as $\vec{f}(\vec{x})$ and $\vec{g}(\vec{x})$ are continuous at $\vec{x}_0$, it follows that $\exists \delta_1, \delta_2 > 0 \ni ||\vec{f}(\vec{x}) - \vec{f}(\vec{x}_0)|| < \frac{\epsilon}{2}$ whenever $||\vec{x} - \vec{x}_0|| < \delta_1$ and $||\vec{g}(\vec{x}) - \vec{g}(\vec{x}_0)|| < \frac{\epsilon}{2}$ whenever $||\vec{x} - \vec{x}_0|| < \delta_2$. Put $\delta_0 : \min\{\delta_1, \delta_2\}$. Then whenever $||\vec{x} - \vec{x}_0|| < \delta_0$, it follows that

$$\begin{aligned} ||(\vec{f}(\vec{x}) + \vec{g}(\vec{x})) - (\vec{f}(\vec{x_0}) + \vec{g}(\vec{x_0})|| &\leq ||\vec{f}(\vec{x}) - \vec{f}(\vec{x_0})|| + ||\vec{g}(\vec{x}) - \vec{g}(\vec{x}_0)|| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Theorem 73: The usual and meaningful combinations of continuous functions are continuous.

Theorem 74: $\vec{F} : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\vec{a} \in \mathbb{X}$ if and only if each component function of $\vec{F}$ is continuous at $\vec{a}$.

Definition 75: $\vec{F} : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is **Lipschitz continuous on** $\mathbb{X}$ provided that there is an $\alpha > 0$ such that

$$||\vec{F}(\vec{x}) - \vec{F}(\vec{y})|| \leq \alpha ||\vec{x} - \vec{y}|| \forall x, y \in \mathbb{X}$$

And notice the similarity to the limit definition: $||\vec{F}(\vec{x}) - \vec{F}(\vec{y})|| < \epsilon$ whenever $||\vec{x} - \vec{y}|| < \delta$.

Remark: If $n = m = 1$, then Definition 75 reduces to

$$|F(x) - F(y)| \leq \alpha |x - y|$$

Remark: If $\alpha \in (0, 1)$, then F is a **contraction mapping**.

Definition 76: Let $f : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}$. Then the **partial derivative of f with respect to x_i** is the (ordinary) derivative of the partial function of f with respect to x_i. That is,

$$\frac{\partial f}{\partial x}(x_0) = f_{x_i} := \lim_{\Delta x_i \to 0} \frac{f(x_1, x_2, ..., x_{i-1}, x_i + \Delta x_i, x_{i+1}, ..., x_n) - f(x_1, x_2, ..., x_n)}{\Delta x_i}$$

where $x_0 := (x_1, ..., x_i, ..., x_n)$.

Example 77:

(a.)

$$f(x, y) = xy^2 + x^2 y$$

$$\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = y^2 + 2xy$$
$$\frac{\partial f}{\partial y}(x, y) = f_y(x, y) = 2xy + x^2$$

(b.)

$$F(x, y, z) = e^{ax} \cos(by) + e^{az} \sin(bx)$$

$$\frac{\partial F}{\partial x} = ae^{ax} \cos(by) + be^{az} \cos(bx)$$
$$\frac{\partial F}{\partial y} = -be^{ax} \sin(by)$$
$$\frac{\partial F}{\partial z} = ae^{az} \sin(bx)$$

Example 78:

$$f(x, y) := \left\{ \begin{array}{lll} \frac{3x^2 y - y^3}{x^2 + y^2} & , & (x, y) \neq (0, 0) \\ 0 & , & (x, y) = (0, 0) \end{array} \right\}$$

Easy to find partials if $(x, y) \neq (0, 0)$. However,

$$\begin{aligned} \frac{\partial f}{\partial x}(0, 0) &= \lim_{\Delta x \to 0} \frac{f(0 + \Delta x, 0) - f(0, 0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{3(\Delta x)^2 (0) - 0^3}{(\Delta x)^2 + 0^2} - 0}{\Delta x} \\ &= 0 \\ \frac{\partial f}{\partial y}(0, 0) &= \lim_{\Delta y \to 0} \frac{f(0, \Delta y + 0) - f(0, 0)}{\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{\frac{3(0)^2 \Delta y - (\Delta y)^3}{0^2 + (\Delta y)^2} - 0}{\Delta y} \\ &= -1 \end{aligned}$$

Example 79: *(A cool example for interpreting the partial derivatives of a function involving a weird college.)*
Platform $9\frac{3}{4}$ (yet again) (Tangency and Differentiability): Recall from Calculus I that if $F : \mathbb{X} \subseteq \mathbb{R} \to \mathbb{R}$ is scalar-valued and $F'(a)$ exists, then the tangent line to F at $x = a$ is $y = F(a) + F'(a)(x - a)$. Put $H(x) := F(a) + F'(a)(x - a)$. Note that

1. $H(a) = F(a)$

2. $H'(a) = F'(a)$

Now consider $F := \mathbb{X} \subseteq \mathbb{R}^2 \to \mathbb{R}$. We want to find a tangent plane to $z = f(x, y)$ at (a,b,f(a,b)). The tangent line in the x-direction is

$$\vec{l}_x(t) = \text{point} + \text{direction vector} = (a, b, f(a, b)) + t(1, 0, f_x(a, b)) \iff \left\{ \begin{array}{l} x = a + t \\ y = b \\ z = f(a, b) + tf_x(a, b) \end{array} \right\}, t \in \mathbb{R}$$

$$\vec{l}_y(t) = (a, b, f(a, b)) + t(0, 1, f_y(a, b))$$

Now cross the direction vectors $\vec{u} = \vec{i} + f_x(a, b)\vec{k}$ and $\vec{v} = \vec{j} + f_y(a, b)\vec{k}$ to get the normal vector

$$\vec{n} = \vec{u} \times \vec{v} = -f_x(a, b)\vec{i} - f_y(a, b)\vec{j} + \vec{k}$$

Therefore the equation of the plane is

$$\begin{aligned} 0 &= \vec{n} \cdot P\vec{P}_0 \\ &= -f_x(a, b)(x - a) - f_y(a, b)(y - b) + z - f(a, b) \\ h(x, y) = z &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \end{aligned}$$

and note that this is analagous to $H(x)$ above.
Theorem 80: If the graph of $z = f(x, y)$ has a tangent plane at $(a, b, F(a, b))$, then the tangent plane has equation

$$z = f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Example 81 (Roof-top Function):

$$f(x, y) := ||x| - |y|| - |x| - |y|$$

$$\begin{aligned} f_x(0,0) &= \lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{|\Delta x| - |\Delta x|}{\Delta x} \\ &= 0 \\ f_y(0,0) &= \lim_{\Delta y \to 0} \frac{f(0, 0+\Delta y) - f(0,0)}{\Delta y} \\ &= \lim_{\Delta x \to 0} \frac{|-|\Delta y|| - |\Delta y|}{\Delta y} \\ &= 0 \end{aligned}$$

So by Theorem 80, the tangent plane to f at $(0,0)$ is

$$h(x,y) := z \equiv 0$$

Now consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - h(x,y)}{||(x,y) - (0,0)||}$$

If this limit exists then the tangent plane approximates the function well.

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{f(x,y) - h(x,y)}{||(x,y) - (0,0)||} &= \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{||(x,y)||} \\ &= \lim_{(x,y)\to(0,0)} \frac{||x| - |y|| - |x| - |y|}{\sqrt{x^2 + y^2}} \end{aligned}$$

Now try to find the limit along two different paths

$$\lim_{x \to 0} \frac{|x| - |x|}{\sqrt{x^2}} = 0$$

$$\begin{aligned} \lim_{y \to 0} \frac{||x| - |x|| - |x| - |x|}{\sqrt{x^2 + x^2}} &= \frac{-2|x|}{\sqrt{2x^2}} \\ &= \lim_{x \to 0} \frac{-2|x|}{|x|\sqrt{2}} \\ &= \frac{-2}{\sqrt{2}} \end{aligned}$$

Therefore the limit does not exist and the tangent plane does not approximate the function well.

Remark: Recall from Calculus I that

$$F'(a) = \lim_{x \to a} \frac{F(x) - F(a)}{x - a}$$

$$\Updownarrow$$

$$\lim_{x \to a} \frac{F(x) - F(a) - (x-a)F'(a)}{x-a} = 0$$
$$\Updownarrow$$
$$\lim_{x \to a} \frac{F(x) - H(x)}{x-a} = 0$$

Definition 82: Let $f : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function. Then the **gradient** of f, denoted ∇f (the **ρ-vector**) or Df (the **matrix of partials** or **Jacobian**) (Okay this looks inconsistent but I still consider a one-by-n matrix a matrix.), is defined to be:

$$\nabla f(\vec{x}_0) := (f_{x_1}(\vec{x}_0), f_{x_2}(\vec{x}_0), ..., f_{x_n}(\vec{x}_0))$$

$$Df(\vec{x}_0) := \begin{bmatrix} f_{x_1}(\vec{x}_0) & f_{x_2}(\vec{x}_0) & \cdots & f_{x_n}(\vec{x}_0) \end{bmatrix}$$

Let $f : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function, say $f(x_1, x_2, ..., x_n) := (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n)$. Then the **matrix of partials of** f, denoted by Df and also known as the **Jacobian**, is defined to be

$$Df(\vec{x}_0) := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}_0) & \frac{\partial f_1}{\partial x_2}(\vec{x}_0) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{x}_0) \\ \frac{\partial f_2}{\partial x_1}(\vec{x}_0) & \frac{\partial f_2}{\partial x_2}(\vec{x}_0) & \cdots & \frac{\partial f_2}{\partial x_n}(\vec{x}_0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{x}_0) & \frac{\partial f_m}{\partial x_2}(\vec{x}_0) & \cdots & \frac{\partial f_m}{\partial x_n}(\vec{x}_0) \end{bmatrix} = \left[\frac{\partial f_i}{\partial x_j}(\vec{x}_0)\right]$$

Definition 83: Let $\mathbb{X} \subseteq \mathbb{R}^n$ be open, let $\vec{f} : \mathbb{X} \to \mathbb{R}^m$, and let $\vec{a} \in \mathbb{X}$. We say that $\vec{f}$ is differentiable at $\vec{a}$ if $D\vec{f}(\vec{a})$ exists and if the function $\vec{h} : \mathbb{R}^n \to \mathbb{R}^m$ defined by $\vec{h}(\vec{x}) := \vec{f}(\vec{a}) + D\vec{f}(\vec{a})(\vec{x} - \vec{a})$ satsfies

$$\begin{aligned} \lim_{\vec{x} \to \vec{a}} \frac{||\vec{f}(\vec{x}) - \vec{h}(\vec{x})||}{||\vec{x} - \vec{a}||} &= \lim_{\vec{x} \to \vec{a}} \frac{||\vec{f}(\vec{x}) - [\vec{f}(\vec{a}) + D\vec{f}(\vec{a})(\vec{x} - \vec{a})]||}{||\vec{x} - \vec{a}||} \\ &= 0 \end{aligned}$$

A better way to do this is to make the limit

$$\lim_{\vec{x} \to \vec{a}} \frac{||\vec{f}(\vec{x}) - \vec{f}(\vec{a}) - \vec{L}(\vec{a})||}{||\vec{x} - \vec{a}||}$$

where $\vec{L}$ is a linear function (maybe).

Theorem 84:

1. If $\vec{f} : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{a}$, then it is continuous at $\vec{a}$.

2. If $\vec{f} : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ such that $\frac{\partial f_i}{\partial x_j}$ exists and is continuous in a neighborhood of $\vec{a}$ in $\mathbb{X}$, for $1 \leq i \leq m$ and $1 \leq j \leq n$, then $\vec{f}$ is differentiable at $\vec{a}$.

3. A function $\vec{f} : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{a} \in \mathbb{X}$ if and only if each of its component functions $f_i : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}$, $1 \leq i \leq m$, is differentiable at $\vec{a}$.

Example 85:

$$f(x,y) := \left\{ \begin{array}{ccc} \frac{-3xy}{x^2+y^2} & , & (x,y) \neq (0,0) \\ 0 & , & (x,y) = (0,0) \end{array} \right\}$$

Along $y = x$

$$\lim_{(x,x)\to(0,0)} \frac{-3x^2}{2x^2} = -\frac{3}{2}$$

But along $y = -x$

$$\lim_{(x,-x)\to(0,0)} \frac{3x^2}{2x^2} = \frac{3}{2}$$

So the function is not continuous, and therefore not differentiable. However, note that

$$\begin{aligned} f_x(0,0) &= \lim_{\Delta x\to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\frac{-3\Delta x(0)}{(\Delta x)^2+0^2} - 0}{\Delta x} \\ &= 0 \\ f_y(0,0) &= \lim_{\Delta y\to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} \\ &= \lim_{\Delta y\to 0} \frac{\frac{-3\Delta y(0)}{0^2+(\Delta y)^2} - 0}{\Delta y} \\ &= 0 \end{aligned}$$

Example 86: Suppose that $\vec{f} : \mathbb{X} \to \mathbb{R}^2$, where $\mathbb{X} := \mathbb{R}^2 \backslash \{(x,y) : x = 0, y = 0\}$,

$$\vec{f}(x,y) = \left(\frac{xy^2}{x^2+y^2}, \frac{x}{y} + \frac{y}{x} \right)$$

$$D\vec{f}(x,y) = \left[\begin{array}{cc} \frac{y^2(x^2+y^4)-(xy^2)(2x)}{(x^6+y^4)^2} & \frac{(2xy)(x^2+y^4)-(xy^2)(4y^3)}{(x^2+y^4)^2} \\ \frac{1}{y} - \frac{y}{x^2} & -\frac{x}{y^2} + \frac{1}{x} \end{array} \right]$$

Proposition 87a: Let $\vec{f}, \vec{g} : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be two functions, each of which is differentiable at $\vec{a} \in \mathbb{X}$, and let $c \in \mathbb{R}$ be any scalar. Then:

1. $\vec{h} := \vec{f} + \vec{g}$ is differentiable at $\vec{a}$ and

$$D\vec{h}(\vec{a}) = D(\vec{f} + \vec{g})(\vec{a}) = D\vec{f}(\vec{a}) + D\vec{g}(\vec{a})$$

2. The function $\vec{k} := c\vec{f}$ is differentiable at $\vec{a}$ and

$$D\vec{k}(\vec{a}) = D(c\vec{f})(\vec{a}) = cD\vec{f}(\vec{a})$$

Proposition 87a: Let $f, g : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\vec{a}\mathbb{X}$. Then

1. The product fg is differentiable at $\vec{a}$ and

$$D(fg)(\vec{a}) = g(\vec{a})Df(\vec{a}) + f(\vec{a})Dg(\vec{a})$$

2. If $g(\vec{a}) \neq 0$, then the quotient $\frac{f}{g}$ is differentiable at $\vec{a}$ and

$$D\left(\frac{f}{g}\right)(\vec{a}) = \frac{g(\vec{a})Df(\vec{a}) - f(\vec{a})Dg(\vec{a})}{(g(x))^2}$$

Example 88:

$$f(x, y) = xy + \cos x$$
$$g(x, y) = \sin(xy) + y^3$$

$$Df(x, y) = \left[\begin{array}{cc} f_x & f_y \end{array}\right] = \left[\begin{array}{cc} y \sin x & x \end{array}\right]$$
$$Dg(x, y) = \left[\begin{array}{cc} g_x & g_y \end{array}\right] = \left[\begin{array}{cc} y \cos(xy) & x \cos(xy) + 3y^2 \end{array}\right]$$

Then

$$h(x, y) := (f + g)(x, y) = xy + y^3 + \cos x + \sin(xy)$$

So

$$\begin{aligned} Dh(x, y) &= Df(x, y) + Dg(x, y) \\ &= \left[\begin{array}{cc} y + y\cos(xy) - \sin x & \text{something} \end{array}\right] \end{aligned}$$

Example 89:

$$f(x, y) = x^3y^7 + 3xy^2 - 7xy$$

First-order partials:

$$\frac{\partial f}{\partial x} = f_x = 3x^2y^7 + 3y^2 - 7y$$
$$\frac{\partial f}{\partial y} = f_y = 7x^3y^6 + 6xy - 7x$$

Second-order partials:

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = 6xy^7$$
$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = 21x^2y^6 + 6y - 7$$
$$\frac{\partial^2 f}{\partial x \partial y} = f_{yx} = 21x^2y^6 + 6y - 7$$
$$\frac{\partial^2 f}{\partial y^2} = f_{yy} = 42x^3y^5 + 6x$$

Some third-order partials:

$$\frac{\partial^3 f}{\partial y \partial x^2} = f_{xxy} = 42xy^6$$
$$\frac{\partial^3 f}{\partial x \partial y \partial x} = f_{xyx} = 42xy^6$$

Platform $9\frac{3}{4}$ (Some Notation): If $f : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ is a scalar-valued function of n variables, then the $k^{\textbf{th}}$**-order partial derivative** with respect to the variables $x_{i_1}, x_{i_2}, ...x_{i_k}$ (in that order), where $i_1, ..., i_k$ are integers from the set $\{1, ..., n\}$, possibly repeated, is the iterated derivative

$$\frac{\partial^k f}{\partial x_{i_k} \partial x_{i_{k-1}} ... \partial x_{i_1}} = \frac{\partial}{\partial x_{i_k}} \frac{\partial}{\partial x_{i_{k-1}}} ... \frac{\partial}{\partial x_{i_1}} (f(x_1, ..., x_n))$$
$$= f_{x_{i_1} x_{i_2} ... x_{i_k}}(x_1, ..., x_n)$$

Remark: Assume $\mathbb{X}$ is open in $\mathbb{R}^n$. A scalar-valued function $f : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}$, whose partials up to and including order at least k exist and are continuous on $\mathbb{X}$ is said to be of class $C^k = C^k(\mathbb{X})$. If partials of all orders exist and ate continuous, then f is of class $C^\infty = C^\infty(\mathbb{X})$ or **smooth**.

Theorem 90 (Claisaut's): Let $f : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function of class C^k. Then the order in which we calculate any $k^{textrmth}$-order partial derivative does *not* matter. That is, if $(i_1, ..., i_k)$ are not necessarily distinct integers between 1 and n, inclusive, and if $(j_1, ..., j_k)$ is any permutation of $(i_1, ..., i_n)$, then

$$\frac{\partial^k f}{\partial x_{i_1} ... \partial x_{i_k}} = \frac{\partial^k f}{\partial x_{j_1} ... \partial x_{j_k}}$$

Definition 91: Let $\mathbb{V}$ be a vector space. Then the function $< \cdot, \cdot >: \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ is an **inner product on** $\mathbb{V}$ if:

1. $< x, x > > 0$ for $x \neq 0$

2. $< x, y >=< y, x >$

3. $< ax + by, z >= a < x, z > +b < y, z >$

Remark:

(A.) (3.) implies $< \vec{0}, \vec{0} >= 0$

(B.) (2.) + (3.) implies that the function is linear in the second variable, too

(C.) Some examples of inner products:

- **(α.)** The dot product.
- **(β.)** The integral $\int_a^b (fg)dx$.

(D.) Given $< \cdot, \cdot >$ on $\mathbb{V}$, we can put $||x|| := \sqrt{< x, x >}$ and this is called the **norm induced by** $< \cdot, \cdot >$.

Definition 92: Let $\mathbb{V}$ be a vector space. Then the function $|| \cdot || : \mathbb{V} \to [0, \infty)$ is called a **norm on** $\mathbb{V}$ provieded that:

1. $||x|| > 0$, unless $x = \vec{0}$
2. $||\alpha x|| = |\alpha| \cdot ||x||, \forall \alpha \in \mathbb{R}$
3. $||x + y|| \leq ||x|| + ||y||$

Remarks:

(A.) By (2.), $||\vec{0}|| = 0$

(B.) Some examples of norms:

- **(α.)** For $\mathbb{V} := \mathbb{R}^n$, $||\vec{x}||$, the Euclidean two-norm
- **(β.)** For $\mathbb{V} := C([a, b])$, $||f||_{L^1} := \int_a^b |f|dx$ and $||f||_{L^p} := \left(\int_a^b |f|dx\right)^{\frac{1}{p}}$

Remark: A sequence $(x_{n_{n=1}^{\infty}})$ is **Cauchy** if $\forall \epsilon > 0, \exists N \in \mathbb{N} \ni |x_n - x_m| < \epsilon$ whenever $n, m \geq N$. A vector space $\mathbb{V}$ is called **complete** is every Cauchy sequence in $\mathbb{V}$ converges to something in $\mathbb{V}$. ...

Definition 93: Let X be a set. Then a **metric on** X is a function ρ such that $\rho : X \times X \to [0, \infty)$ with

1. $\rho(x, y) = 0 \iff x = y$
2. $\rho(x, y) = \rho(y, x), \forall x, y \in X$
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z), \forall x, y, z \in X$

Finally, (X, ρ) is a **metric space**.

Remark: Some examples:

(A.) $(\mathbb{R}^n, \rho)$, where $\rho(x, y) := ||x - y||$

(B.) (X, ρ), where $\rho x, y := \left\{ \begin{array}{ccc} 0 & , & x = y \\ 1 & , & x \neq y \end{array} \right\}$

(C.) $(\mathbb{Z}, \rho)$ where $\rho(n, n) = 0$ and $\rho(m, n) = 2^{-d}$, where d is the greatest power of two dividing $m - n$ (the 2-adic metric)

Definition 94: We say that a set $Y \subseteq X$ where (x, ρ) is a metric space, is totally bounded if for each $\epsilon > 0$, there is a finite open cover of Y of the form $\mathcal{F} := \{\mathcal{B}_\epsilon(y_i)\}_{i=1}^n, y_i \in Y$.

Editor's Note: *Somewhere near this point I gave up trying to consistantly labeling vectors and vector-valued functions with the little arrows.*

Proof (of Theorem 27 (Cauchy-Schwarz)): Given $\vec{x}, \vec{y} \in \mathbb{V}$, put $\vec{u} := \frac{\vec{x}}{||\vec{x}||}$ and $\vec{v} := \frac{\vec{y}}{||\vec{y}||}$. Note that $||\vec{u}|| = ||\vec{v}|| = 1$. Then:

$$\begin{aligned}
0 \leq ||u - v||^2 &= < u - v, u - v > \\
&= < u, u - v > - < v, u - v > \\
&= < u, u > - < u, v > - < v, u > + < v, v > \\
&= -2 < u, v > +2 \\
-2 &= -2 < u, v > \\
1 &\geq < u, v > \\
< u, v > &\leq 1 \\
< \frac{x}{||x||}, \frac{y}{||y||} > &\leq 1 \\
\frac{1}{||x|| \cdot ||y||} < x, y > &\leq 1 \\
< x, y > &\geq ||x|| \cdot ||y|| \\
| < x, y > | \geq ||x|| \cdot ||y||
\end{aligned}$$

Then go back and make $vecr := -\vec{u}$ and do this again, then you can get absolute value.

Corollary 95 (Triangle Inequality): Given $x, y \in \mathbb{R}^n$, $||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$.

Proof:

$$
\begin{aligned}
||\vec{x}+\vec{y}||^2 &= (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) \\
&= \vec{x}\cdot\vec{x} + 2\vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y} \\
&= ||\vec{x}||^2 + 1\vec{x}\cdot\vec{y} + ||\vec{y}||^2 \\
&\leq ||\vec{x}||^2 + 2||\vec{x}||\cdot||\vec{y}|| + ||\vec{y}||^2 \\
&= (||\vec{x}|| + ||\vec{y}||)^2
\end{aligned}
$$

whence

$$||\vec{x}+\vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$$

Definition 96: Let $\mathbb{W}$ and $\mathbb{V}$ be vector spaces. Then $T : \mathbb{V} \to \mathbb{W}$ is linear if and only if

1. $T(x+y) = Tx + Ty, \forall x, y \in \mathbb{V}$

2. $T(\alpha x) = \alpha Tx, \forall \alpha \in \mathbb{R}, x \in \mathbb{V}$

Example 97:

$$T : \mathbb{R}^4 \to \mathbb{R}^3$$

specifically

$$T(x_1, x_2, x_3, x_4) := (x_1 + x_2, x_3 + x_4, x_1 + x_4)$$

which is obviously linear (you can do the math yourself). Let $\alpha = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$ be the standard ordered basis for $\mathbb{R}^4$, and let $\beta = \{(1,0,0),(0,1,0),(0,0,1)\}$ be the standard ordered basis for $\mathbb{R}^3$.

$$
\begin{aligned}
T(1,0,0,0) &= (1,0,1) \\
T(0,1,0,0) &= (1,0,0) \\
T(0,0,1,0) &= (0,1,0) \\
T(0,0,0,1) &= (1,0,0)
\end{aligned}
$$

$$
[T]_\alpha^\beta = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}
$$

Now for example

$$T(1,1,2,0) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$
$$= (2,2,1)$$

Theorem 98: The mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only of there is a matrix A such that $Tx = Ax, \forall x \in \mathbb{R}^n$, where

$$A = [T]_\alpha^\beta = \begin{bmatrix} Te_1 & Te_2 & Te_3 & \cdots & Te_n \end{bmatrix}$$

where α is an ordered basis for $\mathbb{R}^n$ and β is an ordered basis for $\mathbb{R}^m$. (There is something weird about that matrix.)

Proof: The reverse direction is trivially true. To prove the forward direction, let

$$Te_j := \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

and put

$$A := [a_{ij}]$$

where $1 \le i \le m$ and $1 \le j \le n$. So, given any $\vec{x} \in \mathbb{R}^n \ni \vec{x} = (x_1, ..., x_n) = x_1e_1 + ... + x_ne_n$,

$$Tx = T(x_1e_1 + ... + x_ne_n)$$
$$= x_1Te_1 + ... + x_nTe_n$$
$$= x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + ... + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$
$$= \begin{bmatrix} x_1a_{11} + x_2a_{12} + ... + x_na_{1n} \\ \vdots \\ x_1a_{m1} + x_2a_{m2} + ... + x_na_{mn} \end{bmatrix}$$
$$= Ax$$

Remark:

1.

$$T : \mathbb{R}^n \to \mathbb{R}^m$$
$$S : \mathbb{R}^m \to \mathbb{R}^p$$

   Ordered Basis Correspondance:

$$\alpha \to \mathbb{R}^n$$
$$\beta \to \mathbb{R}^m$$
$$\gamma \to \mathbb{R}^p$$
$$[S \cdot T]_\alpha^\gamma = [S]_\beta^\gamma [T]_\alpha^\beta$$

2. The property in (1.) can be used to prove the associativity of matrix multiplication.

3. All of this works in "complete" generality. For example

$$T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}^4$$
$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b, c, d)$$
$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

Recall: $\vec{F} : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\vec{a} \in \mathbb{X}$ if and only if

$$\lim_{\vec{x}\to\vec{a}} \frac{\vec{F}(\vec{x}) - \vec{F}(\vec{a}) - D\vec{F}(\vec{a})(\vec{x} - \vec{a})}{||\vec{x} - \vec{a}||} = 0$$
$$\Updownarrow$$
$$\lim_{\vec{x}\to\vec{a}} \frac{\vec{F}(\vec{x}) - [\vec{F}(\vec{a}) + DF(\vec{a})(\vec{x} - \vec{a})]}{||\vec{x} - \vec{a}||} = 0$$

But $\vec{F}(\vec{a}) + D\vec{F}(\vec{a})(\vec{x} - \vec{a})$ is affine, not linear. But if we set $\Delta\vec{F} = \vec{F}(\vec{x}) = \vec{F}(\vec{a})$ and set $D\vec{F}(\vec{a})(\vec{x} - \vec{a})$ to be a linear transformation L, we have something cooler. What does it mean to say that F is differentiable at $\vec{a}$? Answer: $\vec{F}$ is differentiable at $\vec{a}$ provided that there exists a linear transformation L such that

$$\lim \vec{h} \to \vec{0} \frac{\vec{F}(\vec{a} + \vec{h}) - \vec{F}(\vec{a}) - L(\vec{h})}{||\vec{h}||} = 0$$

and we decide L by $d\vec{F}_{\vec{a}}$, whose matrix is $D\vec{F}(\vec{a})$. $D\vec{F}(\vec{a})$ is the unique $m \times n$ matrix such that $d\vec{F}_{\vec{a}}(\vec{x}) = D\vec{F}(\vec{a})\vec{x}$.

Remark:

$$f(x,y) = x^2 + y^2$$

$$\begin{aligned} \vec{a} &= (2,2) \\ \vec{h} &= (dx, dy) \end{aligned}$$

$$\begin{aligned} \Delta f_{\vec{a}}(\vec{h}) &= F(2+dx, 2+dy) - F(2,2) \\ &= (2+dx)^2 + (2+dy)^2 - 8 \\ &= 4 + 4dx + dx^2 + 4 + 4dy + dy^2 + 4dy \end{aligned}$$

$$\begin{aligned} df_{\vec{a}}(\vec{h}) &= \begin{bmatrix} f_x(\vec{a}) & f_y(\vec{a}) \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \\ &= \nabla f(\vec{a}) \cdot \vec{h} \\ &= f_x(\vec{a})dx + f_{\vec{y}}(\vec{a})dy \\ &= (2x)|_{(2,2)}dx + (2y)|_{(2,2)}dy \\ &= 4dx + 4dy \end{aligned}$$

$$\begin{aligned} \lim_{\vec{h}\to\vec{0}} \frac{(dx^2 + 4dx + dy^2 + 4dy) - (4dx + 4dy)}{\sqrt{dx^2 + dy^2}} &= \lim_{\vec{h}\to\vec{0}} \frac{dx^2 + dy^2}{\sqrt{dx^2 + dy^2}} \\ &= \lim_{\vec{h}\to\vec{0}} \sqrt{dx^2 + dy^2} \\ &= 0 \end{aligned}$$

Example 99: Approximate $\left[(13.1)^2 - (4.9)^2\right]^{1/2}$ by the differential.
Put

$$\begin{aligned} f(x,y) &= \sqrt{x^2 + y^2} \\ \vec{a} &= (13, 5) \\ \vec{h} &= (0.1, -0.1) \end{aligned}$$

$$\begin{aligned} f_x &= \frac{x}{\sqrt{x^2 - y^2}} \\ f_y &= \frac{-y}{\sqrt{x^2 - y^2}} \end{aligned}$$

$$df_{\vec{a}} \approx \Delta f_{\vec{a}}(\vec{h}) = f(15+0.1, 5-0.1) - f(13,5)$$

$$\begin{aligned} f(13.1, 4.9) &\approx \nabla f(\vec{a}) \cdot \vec{h} + f(13,5) \\ &= \begin{bmatrix} \frac{13}{12} & \frac{-5}{12} \end{bmatrix} \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix} + 12 \\ &= \frac{13}{120} + \frac{5}{120} + \frac{1440}{120} \\ &= \frac{1458}{120} \\ &= 12.15 \end{aligned}$$

Theorem 100 (Chain Rule): Let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. If the mapping $F : U \to \mathbb{R}^m$ and $G : V \to \mathbb{R}^k$ are differentiable at $\vec{a} \in U$ and $F(\vec{a}) \in V$, respectively, then $H := G \circ F$ is differentiable at $\vec{a}$ and

1. $dH_{\vec{a}} = dG_{F(\vec{a})} \cdot dF_{\vec{a}}$
2. $DH(\vec{a}) = DG(F(\vec{a}))DF(\vec{a})$

Example 101:

(a.) If $n = m = p = 1$ then

$$\begin{aligned} DH(a) &= [H`(a)] \\ &= DG(F(a))DF(a) \\ &= [G`(F(a))][F`(a)] \\ &= [G`(F(a))F`(a)] \end{aligned}$$

(b.)

$$\begin{aligned} F &: \mathbb{R}^2 \to \mathbb{R}^3 \\ G &: \mathbb{R}^3 \to \mathbb{R}^2 \end{aligned}$$

So

$$\begin{aligned} F(s,t) &= (x,y,z) \\ G(x,y,z) &= (u,v) \end{aligned}$$

$$DF = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{bmatrix}$$

$$DG = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{bmatrix}$$

$$DGDF = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{bmatrix}$$

And you can do the multiplication yourself.

Aside by Guest Lecturer Nick Steinauer (Proof of God's Existence): Notice that God and Gauss both start with the letter G. Also note that

$$\text{God} = \text{Gauss} \tag{1}$$

But Gauss's existence is trivially true. So by Equation (1), God exists.

Example 102:

$$\begin{aligned} f(x, y, z) &= (x + y + z, x^3 - e^{yx}) \\ g(s, t, u) &= (st, tu, su) \\ (f \circ g)(s, t, u) &= (st + tu + su, s^3t^3 - e^{stu^2}) \\ D(f \circ g) &= \begin{bmatrix} t + u & s + u & t + s \\ 3s^2t^3 - tu^2e^{stu^2} & 3t^2s^3 - su^2e^{stu^2} & -2uste^{stu^2} \end{bmatrix} \\ Df(x, y, z) &= \begin{bmatrix} 1 & 1 & 1 \\ 3x^2 & -ze^{yz} & -ye^{yz} \end{bmatrix} \\ Dg(s, t, u) &= \begin{bmatrix} t & s & 0 \\ 0 & u & t \\ u & 0 & s \end{bmatrix} \end{aligned}$$

And when you multiply Dg and Df together and substitute based on the definition of g, you get the same answer.

Example 103 (1-D wave equation):

$$f_{tt} = a^2 f_{xx}$$

Put $x := Au + Bv$ and $t := Cu + Dv$. Write $g(u,v) := f(Au + Bv, Cu + Dv) = f(x,y)$. Now by chain rule

$$\begin{aligned} g_u &= Af_x + Cf_t \\ g_{uv} &= (Af_x)_v + (Cf_t)_v \\ &= ABf_{xx} + ACf_{xt} + CDf_{tx} + BCf_{tt} \\ &= ABf_{xx} + (AD + BC)f_{xt} + CDf_{tt} \end{aligned}$$

Now pick

$$\begin{aligned} A &:= \frac{1}{2} \\ B &:= \frac{1}{2} \\ C &:= \frac{1}{2a} \\ D &:= -\frac{1}{2a} \end{aligned}$$

so that

$$\begin{aligned} g_{uv} &= ABf_{xx} + (AD + BC)f_{xt} + CDf_{tt} \\ &= \frac{1}{4}f_{xx} - \frac{1}{4a^2}f_{tt} \\ &= 0 \end{aligned}$$

So

$$\frac{\partial^2 g}{\partial v \partial u} = 0$$

and

$$g(u,v) := \phi(u) + \psi(v)$$

where $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are arbitrary functions. Now for some more initial conditions.

$$\begin{aligned} f_{tt} &= a^2 f_{xx} \\ f(x,0) &= F(x) \\ f_t(x,0) &= G(x) \end{aligned}$$

So now

$$\begin{aligned} g(u,v) &= \phi(u) + \psi(v) \\ f(x,t) &= \phi(x + at) + \psi(x - at) \end{aligned}$$

From $f(x,0) = F(x)$ we get that

$$f(x,0) = \phi(x) + \psi(x) = F(x)$$

From $f_t(x,0) = G(x)$ we get that

$$f_t(x,0) = a\phi'(x) - a\psi'(x) = G(x)$$

and by the Fundemental Theorem of Calculus

$$a\phi(x) - a\psi(x) = \int_0^x G(s)ds + K$$

Solve this to get

$$\begin{aligned} \phi(x) &= \frac{1}{2}F(x) + \frac{1}{2a}\int_1^x G(s)ds + \frac{K}{2a} \\ \psi(x) &= \frac{1}{2}F(x) - \frac{1}{2a}\int_0^x G(s)ds - \frac{K}{2a} \end{aligned}$$

Thus, substituting $x + at$ for x in $\phi(x)$ and $x - at$ in $\psi(x)$ we get

$$f(x,t) = \frac{F(x+at) - F(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} G(s)ds$$

Example 104: Put

$$\begin{aligned} x &:= r\cos\theta \\ y &:= r\sin\theta \end{aligned}$$

$$\begin{aligned} w &:= g(r,\theta) = f(x(r,\theta), y(r,\theta)) \\ w &:= g(r,\theta) = (f \circ \vec{x})(r,\theta) \end{aligned}$$

$$\begin{aligned} Dg(r,\theta) &= Df(x,y)D\vec{x}(r,\theta) \\ &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\frac{\partial f}{\partial x} & -r\sin\theta\frac{\partial f}{\partial x} \\ \sin\theta\frac{\partial f}{\partial y} & r\cos\theta\frac{\partial f}{\partial y} \end{bmatrix} \end{aligned}$$

Therefore

$$\frac{\partial w}{\partial r} = \cos\theta \frac{\partial w}{\partial x} + \sin\theta \frac{\partial w}{\partial y}$$
$$\frac{\partial w}{\partial \theta} = -r\sin\theta \frac{\partial w}{\partial x} + r\cos\theta \frac{\partial w}{\partial y}$$

And in general

$$\frac{\partial}{\partial r} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial \theta} = -r\sin\theta \frac{\partial}{\partial x} + r\cos\theta \frac{\partial}{\partial y}$$

And in reverse

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} + \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}$$
$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \frac{\partial}{\partial \theta}$$

Proof of Theorem 100 (Chain Rule): Recall that it suffices to show that

$$\lim_{h\to\vec{0}} \frac{H(a+h) - H(a) - dG_{F(a)} \circ dF_a(h)}{||h||} = \vec{0}$$

As F is deifferntiable at a, it follows that

$$\lim_{h\to\vec{0}} \frac{F(a+h) - F(a) - dF_a(h)}{||h||} = \vec{0}$$

and as G is differentiable at $F(a)$, it also follows that

$$\lim_{h\to\vec{0}} \frac{G(F(a)+h) - G(F(a)) - dG_{F(a)}(h)}{||h||} = \vec{0}$$

Put

$$\phi(h) := \lim_{h\to\vec{0}} \frac{F(a+h) - F(a) - dF_a(h)}{||h||}$$

and

$$\psi(k) := \lim_{k\to\vec{0}} \frac{G(F(a)+k) - G(F(a)) - dG_{F(a)}(k)}{||k||}$$

So

$$\begin{aligned}
H(a+h) - H(a) &= G(F(a+h)) - G(F(a)) \\
&= G(F(a) + (F(a+h) - F(a))) - G(F(a)) \\
&= dG_{F(a)}(F(a) + h - F(a)) \\
&+ ||F(a+h) - F(a)|| \frac{G(F(a) + F(a+h) - F(a)) - G(F(a)) - dG_{F(a)}(F(a+h) - F(a))}{||F(a+h) - F(a)||} \\
&= dG_{F(a)}(F(a) + h - F(a)) + ||F(a+h) - F(a)||\psi(F(a+h) - F(a))
\end{aligned}$$

Now

$$\begin{aligned} dF_a(h) + ||h||\phi(h) &= dF_a(h) + F(a+h) - F(a) - dF_a(h) \\ &= F(a+h) - F(a) \end{aligned}$$

So

$$\begin{aligned} H(a+h) - H(a) &= dG_{F(a)}(dF_a(h) + ||h||\phi(h)) + ||F(a+h) - F(a)||\psi(F(a+h) - F(a)) \\ &= dF_{F(a)} \circ dF_a(h) + ||h||dG_{F(a)}(\phi(h)) + ||h|| \left|\left| dF_a\left(\frac{h}{||h||}\right) + \phi(h) \right|\right| \psi(F(a+h) - F(a)) \end{aligned}$$

Note that the first term of this comes from th definition of a linear transform, namely

$$T(x+y) = Tx + Ty$$

and

$$T(\alpha x) = \alpha T x$$

and the second term comes from the fact that

$$||dF_a(h) + ||h||\phi(h)|| = F(a+h) - F(a)$$

Now

$$\lim_{h \to \vec{0}} \frac{H(a+h) - H(a) - dG_{F(a)} \circ dF_a(h)}{||h||} = \lim_{h \to \vec{0}} \left[dG_{F(a)}(\phi(h)) \left|\left| dF_a\left(\frac{h}{||h||}\right) + \phi(h) \right|\right| \psi(F(a+h) - F(a)) \right]$$

Now note the following

1. $\lim_{h\to 0} dG_{F(a)}(\phi(h)) = \vec{0}$
2. $\lim_{h\to 0} \psi(F(a+h) - F(a)) = \vec{0}$
3. $\lim_{h\to 0} \left|\left| dF_a\left(\frac{h}{||h||}\right) + \phi(h) \right|\right| = ||M + \vec{0}|| = \vec{0}$ because $dF_a\left(\frac{h}{||h||}\right)$ is a closed and bounded.

So the whole thing is equal to the $\vec{0}$ and we are done.

Platform $9\frac{3}{4}$ (Intro to Directional Derivatives):

$$\begin{aligned} \frac{\partial f}{\partial x}(a,b) &= \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h} \\ &= \lim_{h\to 0} \frac{f((a,b) + h(1,0)) - f(a,b)}{h} \\ &= \lim_{h\to 0} \frac{f(\vec{a} + h\hat{i}) - f(\vec{a})}{h} \\ &= D_{\hat{i}} f(\vec{a}) \\ &= \nabla f(\vec{a}) \cdot \vec{v} \end{aligned}$$

And change $\hat{\imath}$ to an arbitrary unit vector $\vec{v}$ to get the directional derivative.

Definition 105: Let $\mathbb{X}$ be open in $\mathbb{R}^n$, $f : \mathbb{X} \to \mathbb{R}$ be a scalar-valued function, and $\vec{a} \in \mathbb{X}$. If $\vec{v} \in \mathbb{R}^n$ is any unit vector, then the directional derivative of f at $\vec{a}$ in the direction of $\vec{v}$, denoted $D_{\vec{v}} f(\vec{a})$, is

$$D_{\vec{v}} f(\vec{a}) := \lim_{h \to 0} \frac{f(\vec{a} + h\vec{v}) - f(\vec{a})}{h}$$

provided that the limit exists

Remark: Put $F(t) = f(\vec{a} + t\vec{v})$. Note that by Definition 105,

$$D_{\vec{v}} f(\vec{a}) = \lim_{t \to 0} \frac{F(t) - F(0)}{t - 0} = F'(0)$$

Also then,

$$D_{\vec{v}} f(\vec{a}) = \frac{d}{dt}[f(\vec{a} + t\vec{v})]_{t=0}$$

Now put $\vec{x}(t) = \vec{a} + t\vec{v}$. So $F(t) = (f \circ \vec{x})(t)$. Thus,

$$\begin{aligned}
\frac{d}{dt} f(\vec{a} + t\vec{v}) &= \frac{d}{dt}(F(t)) \\
&= \frac{d}{dt}(f \circ \vec{x})(t) \\
&= Df(\vec{x}) D\vec{x}(t) \\
&= \nabla f(\vec{x}) \cdot D\vec{x}(t)
\end{aligned}$$

Hence,

$$\left[\frac{d}{dt} f(\vec{a} + t\vec{v})\right]_{t=0} = \nabla f(\vec{x}(0)) \cdot D\vec{x}(0) = \nabla f(\vec{a}) \cdot \vec{v}$$

whence

$$D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$$

Theorem 106: Let $\mathbb{X}$ be open and suppose that $f : \mathbb{X} \to \mathbb{R}$ be differentiable at $\vec{a} \in \mathbb{X}$. Then $D_{\vec{v}} f(\vec{a})$ exists for all $\vec{v} \in \mathbb{R}^n$ and $D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$.

Example 107:

$$f(x, y) = 4 - x^2 - \frac{1}{4} y^2$$

Find $D_{\vec{v}}(1, 2)$, when $\vec{v} = \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}$. First check if $||\vec{v}|| = 1$.

$$||\vec{v}|| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$Df(x, y) = \nabla f(x, y) = (f_x, f_y) = (-2x, -\frac{1}{2} y)$$

$$\nabla f(1, 2) = (-2, -1)$$

$$\begin{aligned}\therefore D_{\vec{v}} f(1,2) &= \nabla f(1,2)\cdot \vec{v}\\ &= (-2,-1)\cdot(\frac{1}{2},\frac{\sqrt{3}}{2})\\ &= -1-\frac{\sqrt{3}}{2}\approx 1.866\end{aligned}$$

Remark: Why require $||\vec{v}|| = 1$? If not, say $\vec{w} = k\vec{v}$, for $k \neq 0$. Then

$$\begin{aligned}D_{\vec{w}} f(\vec{a}) &= \lim_{h\to 0}\frac{f(\vec{a}+h\vec{w})-f(\vec{a})}{h}\\ &= \nabla f(\vec{a})\cdot\vec{w}\\ &= \nabla f(\vec{a})\cdot k\vec{v}\\ &= k\nabla f(\vec{a})\cdot\vec{v}\\ &= kD_{\vec{v}} f(\vec{a})\end{aligned}$$

Platform $9\frac{3}{4}$: Note that by Theorem 106, $D_{\vec{v}} f(\vec{a})\nabla f(\vec{a})\cdot\vec{v}$. Recall from THeorem 24 that $\vec{a}\cdot\vec{b} = ||\vec{a}||\cdot||\vec{b}||\cos\theta$. Thus,

$$D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a})\cdot\vec{v} = ||\nabla f(\vec{a})||\cdot||\vec{v}||\cos\theta = ||\nabla f(\vec{a})||\cos\theta$$

Therefore,

$$-||\nabla f(\vec{a})|| \leq D_{\vec{v}} \leq ||\nabla f(\vec{a})||$$

Moreover, it is clear that this is the best possible bound (i.e., it is sharp).

Theorem 108: $D_{\vec{u}} f(\vec{a})$ is maximized with respect to direction when $\vec{u}$ points in the same direction as $\nabla f(\vec{a})$ and minimized when $\vec{u}$ has direction opposite of $\nabla f(\vec{a})$. Furthermore, $\max_{\vec{u}} D_{\vec{u}} f(\vec{a}) = ||\nabla f(\vec{a})||$ and $\min_{\vec{u}} D_{\vec{u}} f(\vec{a}) = -||\nabla f(\vec{a})||$.

Example 109 [number 11, pg 168]:

$$D(x,y) = 400 - 3x^2y^2$$

$$\begin{aligned}\nabla D(x,y) &= (-6xy^2, -6x^2y)\\ \nabla D(1,2) &= (-24,12)\end{aligned}$$

Put

$$\vec{u} := \frac{\nabla D(1,-2)}{||\nabla D(1,-2)||} = \frac{(-24,12)}{\sqrt{720}} = \frac{-2\vec{i}+\vec{j}}{\sqrt{5}}$$

And now

$$D_{\vec{u}} D(1,-2) = ||\nabla D(1,-2)|| = 12\sqrt{5}\frac{\text{ft}}{\text{m}}$$

Theorem 110: Let $\mathbb{X} \subseteq \mathbb{R}^n$ be open and $f : \mathbb{X} \to \mathbb{R}$ be a function of class C^1. If $\vec{x}_0$ is a point on the level set $S := \{\vec{x} \in \mathbb{X} : f(\vec{x}) = c\}$, then $\nabla f(\vec{x}_0) \perp S$

Proof: Let Γ be given parametrically by $\vec{x}(t) = (x_1(t), ..., x_n(t))$, where $a < b < t$ and $\vec{x}(t_0) = \vec{x}_0$ (so, $\vec{x}'(t_0) = \vec{v}$). Since $\Gamma \in S$, we have

$$f(\vec{x}(t)) = f(x_1(t), ..., x_n(t)) = c$$

thus

$$\frac{d}{dt}[f(\vec{x}(t))] = \frac{d}{dt}[c] = 0$$
$$\frac{d}{dt}[f(\vec{x}(t))] = \nabla f(\vec{x}(t_0)) \cdot \vec{x}'(t)$$

But then

$$\nabla f(\vec{x}(t_0)) \cdot \vec{x}'(t) = \nabla f(\vec{x}(t_0)) \cdot \vec{v} = 0$$

Platform $9\frac{3}{4}$ (Return to Tangent Planes): In gereral, if S is a surface defined by the equation $f(x, y, z) = c$, then if $\vec{x}_0 \in S$, then $\nabla f(\vec{x}_0) \perp S$, and so $\nabla f(\vec{x}_0) \cdot (\vec{x} - \vec{x}_0) = 0$ or in other words

$$0 = f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$$

Example 111:

$$x^3 + y^3 + z^3 = 7$$
$$x_0 = (0, -1, 2)$$

Put $F(x, y, z) = x^3 + y^3 + z^3$. Then

$$F_x = 3x^2$$
$$F_y = 3y^2$$
$$F_z = 3z^2$$
$$\nabla F(\vec{x}) = (3x^2, 3y^2, 3z^2)$$
$$\nabla F(0, -1, 2) = (0, 3, 12)$$

Therefore the equation of the plane is

$$(0, 3, 12) \cdot (x - 0, y + 1, z - 2) = 0$$
$$3(y + 1) + 12(z - 2) = 0$$

Remark: Sometimes this fails.

Remark: Let $z = f(x, y)$ be differentiable. Recall that the tangent plane to the graph of z at point (a, b) is given by

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Note that if we put $F(x, y, z) := f(x, y) - z = 0$, then $\nabla F(x, y, z) = (f_x(x, y), f_y(x, y), -1)$

$$\therefore \nabla F(a, b, f(a, b)) \cdot (x - a, y - b, z - f(a, b)) = 0$$
$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - z + f(a, b) = 0$$

so

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

and of course all of this works in n dimensions.

Example 112:

$$F(x_1, ..., x_5) := \sin x_1 + \cos x_2 + \sin x_3 + \cos x_4 + \sin x_5 = -1$$
$$\nabla F(x_1, ..., x_5) = (\cos x_1, -\sin x_2, \cos x_3, -\sin x_4, \cos x_5)$$
$$\nabla F(\pi, \pi, \frac{3\pi}{2}, 2\pi, 2\pi) = (-1, 0, 0, 0, 1)$$

Therefore the equation of the taangent plane is

$$\nabla F(\pi, \pi, \frac{3\pi}{2}, 2\pi, 2\pi) \cdot (x_1 - \pi, x_2 - \pi, x_3 - \frac{3\pi}{2}, x_4 - 2\pi, x_5 - 2\pi) = 0$$
$$(-1)(x_1 - \pi) + 1(x_5 - 2\pi) = 0$$
$$-x_1 + x_5 = \pi$$

Example 113:

$$tu_{xx} - 4u_x = 0$$

Where $u = u(x, t)$. Put $v = u_x$. Then $tv_x - 4v = 0 \iff tv' - 4v = 0 \iff v' = \frac{4}{t}v$. Now

$$\int \frac{1}{v} dv = \int \frac{4}{t} dx$$
$$\ln v = \frac{4}{t}x + \alpha(t)$$
$$v(x) = e^{\frac{4}{t}x} e^{\alpha(t)} = e^{\frac{4}{t}x} \beta t$$

Where $\beta(t) = e^{\alpha(t)}$. Now

$$u_x = e^{\frac{4}{t}x} \beta t$$
$$u = \int e^{\frac{4}{t}x} \beta t \partial x + \gamma(t)$$
$$u(x, t) = \frac{1}{4} \beta(t) e^{\frac{4x}{t}} + \gamma(t)$$

Example 114: Consider $u_t = ku_{xx}$, the one-dimensional diffusion equation. Let's find the caloric equation that satisfies our PDE. Find a solution of the form $u(x,t) = U(z)$ where $z = \frac{x}{\sqrt{kt}}$, k constant. Find u_t in terms of U:

$$u(t) = U'(z) \cdot \frac{\partial z}{\partial t} = U'(z) \cdot \frac{-xk}{2kt\sqrt{kt}} = -\frac{xU'(z)}{2t\sqrt{kt}}$$

Now by chain rule:

$$\begin{aligned} u_x &= U'(z)u_x \\ u_{xx} &= U'' u_x u_x + U'(t)u_{xx} \end{aligned}$$

So

$$\begin{aligned} u_{xx} &= U'(z)\frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x}\right)^2 U''(z) \\ &= U'(z) \cdot 0 + \frac{1}{kt} \cdot U''(z) \\ &= \frac{1}{kt}U''(z) \end{aligned}$$

$$\begin{aligned} \therefore \frac{-xU'(z)}{2t\sqrt{kt}} &= k\frac{1}{kt}U''(z) \\ U''(z) + \frac{x}{2\sqrt{kt}}U'(z) &= 0 \\ U''(z) + \frac{1}{2}zU'(z) &= 0 \end{aligned}$$

Definition 115 (Path in $\mathbb{R}^n$): A **path in** $\mathbb{R}^n$ is a continuous function $\vec{x} : I \subseteq [a,b] \to \mathbb{R}^n$, where $\vec{x}(a)$ and $\vec{x}(b)$ are called the **endpoints** of the path $\vec{x}$. (Note that $[a,b]$ can be replaced by a more general set.)
Example 116:

$$\begin{aligned} \vec{y} &: [0, 2\pi) \to \mathbb{R}^3 \\ \vec{y}(t) &:= (2\cos t, 2\sin t, t) \end{aligned}$$

Let $\vec{\tilde{y}} := (2\cos t, 2\sin t, 0)$, which is the projection of $\vec{y}$ onto $\mathbb{R}^2$. Then $||\vec{\tilde{y}}|| = 2$ and it is a circle. Now put back the z-coordinate and we get a spiral helix thing which everyone has heard about because of deoxyribonucleic acid.
Remark: What should $\vec{x}'(t)$ be?

$$\vec{x}(t) = (x_1(t), ..., x_n(t))$$

$$D\vec{x}(t) = \vec{x}'(t) = \begin{bmatrix} \vec{x}_1'(t) \\ \vdots \\ \vec{x}_n'(t) \end{bmatrix}$$

Now back to the example.

$$\vec{y}' = (-2\sin t, -2\cos t, 1)$$
$$\vec{y}'' = (-2\cos t, -2\sin t, 0)$$

And note that $||\vec{y}''|| \equiv 2$.

Proposition 117: Let $\vec{x}$ be a differentiable path and assume that $\vec{v}_0 = \vec{v}(t_0) \neq 0$. Then a vector parametric equation for the line tangent to $\vec{x}$ at $\vec{x}_0$ is either

$$\vec{l}(s) = \vec{x}_0 + s\vec{v}_0$$

or

$$\vec{v}(t) = \vec{x}_0 + (t - t_0)$$

Definition 118: A **vector field on** $\mathbb{R}^n$ is a mapping of the sort $F : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$.

Example 119:

$$\vec{F}(x, y) := \vec{i} + \vec{j}$$

There was a picture of a really boring vector field here.

Example 120:

$$\vec{F}(x, y) := (y, -x)$$

Hey it looks like a swastika. Moving on then. Note that because of the way the vector field works, all inputs on a circle of radius r have magnitude r and are tangent to the circle pointing in the counterclockwise or, as you may prefer, anticlockwise direction.

Definition 121: A **gradient field on** $\mathbb{R}^n$ or **conservative vector field** is a vector field $\vec{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ such that there exists a scalar function $f : X\mathbb{R}$ so that $\vec{F}(\vec{x}) = \nabla f(\vec{x}), \forall \vec{x} \in X$. In this case, $f(\vec{x})$ is called a **potential function**. (We will find out how to actually do this in Section 6.3.)

Definition 122: A **flow line** of a vector field $\vec{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable path $\vec{x} : I \to X \ni \vec{x}'(t) = \vec{F}(\vec{x}(t))$.

$$\vec{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} F_1(x_1(t), x_2(t), ..., x_n(t)) \\ F_2(x_1(t), x_2(t), ..., x_n(t)) \\ \vdots \\ F_n(x_1(t), x_2(t), ..., x_n(t)) \end{bmatrix} = \vec{F}(\vec{x}(t))$$

$$x_1'(t) = F_1(x_1(t), x_2(t), ..., x_n(t))$$
$$x_2'(t) = F_2(x_1(t), x_2(t), ..., x_n(t))$$
$$\vdots = \vdots$$
$$x_n'(t) = F_n(x_1(t), x_2(t), ..., x_n(t))$$

Which is an n-dimensional system of ODEs.

Example 123: Back to the vector field $F(x,y) = (y, -x)$. Now let's find the flow lines. By Definition 122 we need to solve:

$$x'(t) = y$$
$$y'(t) = -x$$

Now look at it and see that $\ddot{x} = \dot{y} = -x$. This means that $\ddot{x} + x = 0$. Now for our ansatz: we think the answer will be of the form $x(t) := c_1 \cos t + c_2 \sin t$ because then $\ddot{x} + x = 0$. Thus $y(t) = -\dot{x} = -c_1 \sin t + c_2 \cos t$ Now for some initial conditions:

$$x(0) = 0$$
$$y(0) = 1$$

To satisfy these we want

$$c_1 = 0$$
$$c_2 = 1$$

So that

$$x(t) = \sin t$$
$$y(t) = \cos t$$

And we are done. Now to find a function such that given initial conditions it returns the flow line as a function of t. To do this, put $c_1 = x_0$ and $c_2 = y_0$ and treat these as constants. Now we have

$$x_0 = x \cos t_0 + y \sin t_0$$
$$y_0 = -x \sin t_0 + y \cos t_0$$

So now given the initial positions x_0 and y_0 at the initial time t_0,

$$x(t) = x_0 \cos t_0 + y_0 \sin t_0$$
$$y(t) = -x_0 \sin t_0 + y_0 \cos t_0$$

Thus the flow of the vector field is

$$\vec{\phi}(t, x, y) = \vec{\phi}(t, \vec{x}) = (x \cos t + y \sin t, -x \sin t + y \cos t)$$

Where t is the time you want and $\vec{x}$ contains your initial conditions. And note that $\vec{\phi} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$. (It maps time by space to space.)

Example 124:

$$\vec{F}(x, y) = (y, x) \iff \left\{ \begin{array}{c} x'(t) = y \\ y'(t) = x \end{array} \right\}$$

We know that $\dot{x} = y$, $\dot{y} = x$, and $\ddot{x} = \dot{y}$, so $\ddot{x} - x = 0$. So our ansatz suggests that the solution will have the form $x(t) = e^{at}$. So

$$\begin{aligned} a^2 e^{at} - e^{at} &= 0 \\ a^2 - 1 &= 0 \\ a &= \pm 1 \end{aligned}$$

Thus e^t, e^{-t} are solutions. So the general solution is

$$\begin{aligned} x(t) &= c_1 e^t + c_2 e^{-t} \\ y(t) &= c_1 e^t - c_2 e^{-t} \end{aligned}$$

Now to find the flow. At time $t = 0$, $x = x_0$ and $y = y_0$ so

$$\begin{aligned} x_0 &= c_1 + c_2 \\ y_0 &= c_1 - c_2 \end{aligned}$$

Solving for c_1 and c_2 we get

$$\begin{aligned} c_1 &= \frac{x_0 + y_0}{2} \\ c_2 &= \frac{x_0 - y_0}{2} \end{aligned}$$

And thus the flow of the vector field is

$$\vec{\phi}(t, x, y) = \left(\left(\frac{1}{2}x + \frac{1}{2}y \right) e^t + \left(\frac{1}{2}x - \frac{1}{2}y \right) e^{-t}, \left(\frac{1}{2}x + \frac{1}{2}y \right) e^t - \left(\frac{1}{2}x - \frac{1}{2}y \right) e^{-t} \right)$$

Definition 125: Let $\vec{F}$ be a vector field on some subset of $\mathbb{R}^n$. The **flow** of $\vec{F}$ (the flow of $\vec{x}' = \vec{F}(\vec{x})$) is the function $\vec{\phi}(t, \vec{x})$ satisfying:

1. $t \longmapsto \vec{\phi}(t, \vec{x})$ is a solution of the ODE $\vec{x}' = \vec{F}(\vec{x})$ for each fixed $\vec{x}$

2. $\phi(0, \vec{x}) = \vec{x}$

Theorem 126: If $\vec{F}(\vec{x})$ is Lipschitz, then $\phi_{\vec{F}}(t, \vec{x})$ is continuous in all veriables.
Proposition 127: The flow of the vector field $\vec{F}(\vec{x})$ satisfies the following whenever the indicated flows exist:

$$\phi_{\vec{F}}(t_1, \phi_{\vec{F}}(t_0, \vec{x})) = \phi_{\vec{F}}(t_0 + t_1, \vec{x})$$

Definition 128: Let X and Y be metric spaces. A mapping $f : X \to Y$ is a **homeomorphism** if

1. f is continuous
2. f is bijective (so f^{-1} exists)
3. f^{-1} is continuous

Example 129:

$$h(x, y) = (x, \sqrt[3]{y}), h : \mathbb{R}^2 \to \mathbb{R}^2$$

$$(x_1, \sqrt[3]{y_1}) = (x_2, \sqrt[3]{y_2}) \iff x_1 = x_2 \text{ and } y_1 = y_2$$

$$(x_1, y_1) \longmapsto (x_1, \sqrt[3]{y_1})$$

So

$$h^{-1}(x, y) = (x, y^3)$$

And by the above all three conditions are satisfied and h is a homeomorphism.
Definition 130: Suppose that $\vec{F_1}$ and $\vec{F_2}$ are vector fields defined, respectively, on Ω_1, $\Omega_2 \subseteq \mathbb{R}^n$. A homeomorphism h is called a **topological equivalence** of $\vec{F_1}$ and $\vec{F_2}$ if h sends oriented trajectories of $\vec{x}' = \vec{F_1}(\vec{x})$ to oriented trajectories of $\vec{x}' = \vec{F_2}(\vec{x})$.
Definition 131: The homeomorphism h is a **topological conjugacy** of the flows $\phi_{\vec{F_1}}(t, \vec{x})$ and $\phi_{\vec{F_2}}(t, \vec{x})$ if

$$h(\phi_{\vec{F_1}}(t, \vec{x})) = \phi_{\vec{F_2}}(t, h(\vec{x}))$$

Definition 132: The homeomorphism h is a **differential conjugacy** of the vector fields $\vec{F_1}(\vec{x})$ and $\vec{F_2}(\vec{x})$ if h and h^{-1} are differentiable and

$$d_{\vec{x}}h(f_1(\vec{x})) = f_2(h(\vec{x}))$$

Example 133:

$$\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2, \vec{F}(x, y) = (x, y)$$
$$\vec{G} : \mathbb{R}^2 \to \mathbb{R}^2, \vec{G}(x, y) = (x + 1, y + 1)$$

For $\vec{F}$, $x' = x$ and $y' = y$ so

$$\phi_{\vec{F}}(t, x, y) = (xe^t, ye^t)$$

Likewise for $\vec{G}$, $x' = x+1$ and $y' = y+1$ so

$$\phi_{\vec{G}}(t,x,y) = ((x+1)e^t - 1, (y+1)e^t - 1)$$

Now consider the function

$$\vec{h}(x,y) := (x-1, y-1)$$

Note that $\vec{h}$ is a contiuously invertable bijection and therefore a homeomorphism. So now consider

$$\begin{aligned} h(\phi_{\vec{F}})(t,\vec{x}) &= h(xe^t - 1, ye^t - 1) \\ \phi_{\vec{G}}(t, h(\vec{x})) &= \phi_{\vec{G}}(t, x-1, y-1) \\ &= ((x-1+1)e^t - 1, (y-1+1)e^t - 1) \\ &= (xe^t - 1, ye^t - 1) \end{aligned}$$

Therefore $\vec{F}$ and $\vec{G}$ are topologically conjugate. (Thus the trajectories of $\vec{F}$ can be "stretched and superimposed" on the trajectories of $\vec{G}$ such that the parametrizations coincide.) Also note that since $\vec{h}$ is a diffeomorphism and considering

$$\begin{aligned} D\vec{h}(\vec{F}(\vec{x})) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} (x-1)+1 \\ (y-1)+1 \end{bmatrix} \\ &= \vec{G}(\vec{h}(\vec{x})) \end{aligned}$$

And therefore $\vec{F}$ and $\vec{G}$ are differnetially conjugate, which has applications as a change of variables equation.
Definition 134: In $\mathbb{R}^n$, the ∇ (**del** or **gradient**) operator is

$$\nabla := \frac{\partial}{\partial x_1}\vec{e}_1 + \frac{\partial}{\partial x_2}\vec{e}_2 + ... + \frac{\partial}{\partial x_n}\vec{e}_n = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\vec{e}_i$$

Defintition 135: Let $\vec{F} : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable vector field. Then the **divergence** of $\vec{F}$, denoted $\nabla \cdot \vec{F}$ or div $\vec{F}$, is the scalar field

$$\text{div } \vec{F} = \nabla \cdot \vec{F} := \frac{\partial F_1}{x_1} + ... + \frac{\partial F_n}{x_n}$$

Remark:

$$\text{div } \vec{F} = \text{tr}(D\vec{F})$$

Example 136:

$$\begin{aligned}\vec{F} &= {x_1}^2\vec{e}_1 + 2{x_2}^2\vec{e}_2 + ... + n{x_n}^2\vec{e}_n \\ &= ({x_1}^2, 2{x_2}^2, ..., n{x_n}^2)\end{aligned}$$

$$\begin{aligned}\therefore \text{div } \vec{F} = \nabla \cdot \vec{F} &= \frac{\partial F_1}{x_1} + ... + \frac{\partial F_n}{x_n} \\ &= 2x_1 + 4x_2 + ... + 2nx_n\end{aligned}$$

Remark: div $\vec{F}$ at x_0 is a measure of the **net mass flow** or **flux density** of $\vec{F}$ out of x_0.
Remark: If $\vec{F}$ is a velocity field, then

1. $\nabla \cdot \vec{F} = 0 \Rightarrow$ rate of fluid in at $x_0 =$ rate of fluid out at x_0 and $\vec{F}$ is a divergence free field is this holds at each $x_0 \in \mathbb{R}$, which has to do with incompressibility and solenoidal things

2. $\nabla \cdot \vec{F} > 0 \Rightarrow$ more flowing in than out

3. $\nabla \cdot \vec{F} < 0 \Rightarrow$ more flowing out than in

Definition 137: Let $\vec{F} : X \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable vector field. Then the **curl** of $\vec{F}$, denoted $\nabla \times \vec{F}$ or curl $\vec{F}$ is the vector field given by

$$\begin{aligned}\nabla \times \vec{F} :&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}\end{aligned}$$

Example 138:

$$\vec{F} = x^2\vec{i} - e^y\vec{j} + 2xyz\vec{k}$$

$$\begin{aligned}\nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -e^y x & 2xyz \end{vmatrix} \\ &= 2xz\vec{i} + 0\vec{j} - e^y\vec{k} + 0\vec{i} - 2yz\vec{j} - 0\vec{k} \\ &= 2xz\vec{i} - 2yz\vec{j} - e^y\vec{k}\end{aligned}$$

Definition 139: A vector field is **irrotational** is $\nabla \times \vec{F} \equiv 0$.
Theorem 140: Let $f : X \subseteq \mathbb{R}^3 \to \mathbb{R}$ be of class C^2. Then $\nabla \times (\nabla f) \equiv 0$. So, gradient fields are irrotational.

Theorem 141: Let $\vec{F} : X \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field of class C^2. Then $\nabla \cdot (\nabla \times \vec{F}) \equiv 0$. Thus, the curl of $\vec{F}$ is an incompressible vector field.

Theorem 142 (Taylor's Theorem in Several Variables): Let $X \subseteq \mathbb{R}^n$ be open and let $f : X \to \mathbb{R}$ be differentiable at $\vec{a} \in X$. Let $p_1(\vec{x}) = f(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$. Then $f(\vec{x}) = p_1(\vec{x}) + R_1(\vec{x}, \vec{a})$, where

$$\frac{R_1(\vec{x}, \vec{a})}{||\vec{x} - \vec{a}||} \to 0$$

as $\vec{x} \to \vec{a}$.

Reminders: The **total differential** of f is

$$df_{\vec{a}}(\vec{h}) = df(\vec{a}, \vec{h}) = \nabla f(\vec{a}) \cdot \vec{h}$$

since

$$\delta f_{\vec{a}}(\vec{h}) = f(\vec{a} + \vec{h}) \cdot f(\vec{a})$$

and now back to

Taylor's Theorem in Several Variable at Second Degree:

$$p_2(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

and we want all possible first and second order partials of p to equal all possible first and second order partials of f, respectively. The only way to get this is

$$p_2(x) = f(a) + \sum_{i=1}^{n} f_{x_i}(a)(x_i - a_i) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} f_{x_i x_j}(a)(x_i - a_i)(x_j - a_j)$$

Definition 143: Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function. Then the **Hessian** of f, denoted by Hf or $D^2 f$, is the matrix

$$[Hf]_{ij} = [D^2 f]_{ij} = f_{x_i x_j}$$

and no I'm not going to spell it out for you.

Remark: $\mathrm{tr}(Hf) = \Delta f$

Rewrite of Taylor's Theorem in Several Variable at Second Degree With the Hessian: Put $h = x - a$. Then

$$\begin{aligned} p_2(x) &= f(a) + Df(a) \cdot h + \frac{1}{2}\begin{bmatrix} h_1 & \dots & h_n \end{bmatrix} \begin{bmatrix} f_{x_1x_1}(a) & \dots & f_{x_1x_n}(a) \\ \vdots & \ddots & \vdots \\ f_{x_nx_1}(a) & \dots & f_{x_nx_n}(a) \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} \\ &= f(a) + Df(a) \cdot h + h^{\mathrm{T}} Hf(a) h \end{aligned}$$

Definition 144: Given $A = [a_{ij}]$, $A^{\mathrm{T}} = [a_{ji}]$.

Definition 145: Let X be open in $\mathbb{R}^n$ and $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$. We say that f has a **local minimum** at $a \in X$ if $\exists \epsilon > 0 \ni f(x) \geq f(a) \forall x \in \mathcal{B}_\epsilon(a)$. Similarly, f has a **local maximum** at $a \in X$ if $\exists \epsilon > 0 \ni f(x) \leq f(a) \forall x \in \mathcal{B}_\epsilon(a)$.

Definition 146: A point $a \in X$, where $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$, is called a **critical point** of f if $\nabla f(a) = \vec{0}$.

Theorem 147: Let X be open in $\mathbb{R}^n$ and $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$. If f has a local extremum at $a \in X$, then $Df(a) = 0$.

Example 148:

$$f(x, y) = x^2 - y^2$$
$$Df(x) = \begin{bmatrix} 2x & -2y \end{bmatrix}$$

Therefore the only critical point is $x = \vec{0}$. But notice that it is neither a minimum nor a maximum, rather it exhibits a mixed behavior which makes it a saddle point.

Example 149:

$$f(x, y) = 4x + 6y - 12 - x^2 y^2$$
$$Df(x, y) = \begin{bmatrix} 4 - 2x & 6 - 2y \end{bmatrix}$$

So $(2, 3)$ is the only critical point of f. Now looking at the increment of f, we can see that $(2, 3)$ is a local maximum.

Theorem 150 (Second Derivative Test): Given a critical point a of a function f of class C^2, consider the Hessian of f at a. Calculate the sequence of principal minors of $Hf(a)$, which is the sequence $\{d_k\}_{k=1}^n$, where $d_k :=$ $\det H_k$, where H_k is the upper-leftmost $k \times k$ submatrix of $Hf(a)$. Then the following holds:

1. $d_k > 0 \forall k \Rightarrow f$ has a local minimum at a
2. $d_k < 0$ for odd k and $d_k > 0$ for even $k \Rightarrow f$ has a local maximum at a
3. If neither case above applies but $\det Hf(a) \neq 0$, then a is a saddlw point.
4. If $\det Hf(a) = 0$, then the test fails and we call a degenerate.

Example 151: Let $f(x, y, z) = x^2 + y^2 + 2z^2 + xz$. Then

$$Df(x, y, z) = \begin{bmatrix} 2x + z & 2y & 4z + x \end{bmatrix}$$

Therefore the only critical point is (0,0,0). Now

$$Hf(x, y, z) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

Thus we can look at the sequence of principal minors

$$d_1 = 2$$
$$d_2 = 4$$
$$d_3 = 14$$

Therefore f has a relative minimum at $x = a$.

Theorem 152 (Extreme Value Theorem): If $X \subseteq \mathbb{R}^n$ is compact and f is continuous, then f attains both its absolute maximium and absolute minimum on its domain.

Example 153: Let $f(x, y) = x^2 + xy + y^2 - 6y$, $[-3, 3] \times [0, 5] \to \mathbb{R}$. Along the boundary $x = 3$, $f(x, y)$ reduces to $g_1(y) = f(3, y) = y^2 - 3y + 9$. Since $g_1'(y) = 2y - 3$, the f could have an extremum at (3,0), (3,5), $(3, \frac{3}{2})$. Repeat for every boundary and the interior to find the absolute maximum.

Example 158:

$$f(x, y) = 5x + 2y$$

Constraint:

$$5x^2 + 2y^2 = 14$$

Put $g(x) = 5x^2 + 2y^2 = 14$. So the constraint is a level surface of $g(x, y)$.

$$\nabla f(x, y) = (5, 2)$$
$$\nabla g(x, y) = (10x, 4y)$$

$$5 = 10\lambda x$$
$$2 = 4\lambda y$$
$$5x^2 + 2y^2 = 14$$

because $\nabla f = \lambda \nabla g$. Solving the system (which is easy), we quickly see that

$$x = \frac{1}{2\lambda}$$
$$y = \frac{1}{2\lambda}$$

Solving for λ we get

$$\lambda = \pm \frac{1}{2\sqrt{2}}$$

and therefore

$$(x, y) = \left\{ \left(\sqrt{2}, \sqrt{2}\right), \left(-\sqrt{2}, -\sqrt{2}\right) \right\}$$

So if there is a constrained extremum these are the only places it could happen.

Theorem 159 (Multiple Constraints): Let X be open in $\mathbb{R}^n$ and let $f, g_1, g_2, ..., g_k : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be C^1 functions where $k < n$. Let $S = \{\vec{x} \in X : g_1(\vec{x}) = c_1, ...g_k(\vec{x}) = c_k\}$. If $f|_S$ has an extremum at $\vec{x}_0 \in S$, where $\{\nabla g_1(\vec{x}_0), ..., \nabla g_k(\vec{x}_0)\}$ is a linearly independent set, then $\exists \lambda_1, ..., \lambda_k \in \mathbb{R} \ni$

$$\nabla f(\vec{x})_0 = \lambda_1 \nabla g_1(\vec{x}_0) + ... + \lambda_k \nabla g_k(\vec{x}_0)$$

Example 160:

$$f(x, y, z) = 2x + y^2 - z^2$$

Constraints:

$$\begin{aligned} x - 2y &= 0 \\ x + z &= 0 \end{aligned}$$

Thus

$$\begin{aligned} \nabla f(x, y, z) &= (2, 2y, -2z) \\ \nabla g_1(x, y, z) &= (1, -2, 0) \\ \nabla g_2(x, y, z) &= (1, 0, 1) \end{aligned}$$

We want

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$$

So our system is

$$\begin{aligned} 2 &= \lambda_1 + \lambda_2 \\ 2y &= -2\lambda \\ -2z &= \lambda_2 \\ x - 2y &= 0 \\ x + z &= 0 \end{aligned}$$

We quickly find that

$$2 = -y - 2z$$

which, together with the last 2 equations form and easily solvable system.

$$(x, y, z) = \left(\frac{4}{3}, \frac{2}{3}, -\frac{4}{3}\right)$$

So if there is a constrained extremum this is the only place it could happen.

Proposition 161: Let R be a rectangle $R := \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d\}$. Let f be continuous and nonnegative on R. Then the volume V under the graph of f and above R is

$$V := \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Definition 162: Given a closed rectangle $R := [a,b] \times [c,d]$, a **partition of** R consists of a collection of **partition points** that break up or subdivide R into a union of n^2 subrectangles.

Remarks:

$$a = x_0 < x_1 < ... < x_n = b$$
$$c = y_0 < y_1 < ... < y_n = d$$

$$\{x_n\}_{n=0}^{\infty}$$
$$\{y_n\}_{n=0}^{\infty}$$

$$P_x = \{x_0, ..., x_n\}$$
$$P_y = \{y_0, ..., y_n\}$$

$$\Delta x_i := x_i - x_{i-1}, 1 \leq i \leq n$$
$$\Delta y_i := y_i - y_{i-1}, 1 \leq i \leq n$$

Definition 163: Let f be aby funciton defined on $R = [a,b] \times [c,d]$, partition R in some way. Let $\vec{c}_{ij}$ be any point in the ij^{th} subrectangle, that is, $\vec{c}_{ij} \in [x_{i-1}, x_i] \times [y_{i-1}, y_i]$ for $1 \leq i,j \leq n$. Then $S := \sum_{n=1}^{n} \sum_{j=1}^{n} f(\vec{c}_{ij})\Delta A_{ij}$ where ΔA_{ij} where $\Delta A_{ij} = \Delta x_i \Delta y_j$ which is the area of the ij^{th} subrectangle, is called the **Riemann sum** of f on R corresponding to the partition.

Definition 164: The **double integral** of f over R denoted $\int\int_R f dA$ or $\int\int_R f(x,y)dA$ or $\int\int_R f(x,y)dxdy$, is defined by

$$\int\int_R f dA := \lim_{\max \Delta x_i, \Delta y_j \to 0} \sum_{i=1}^{n} \sum_{j=1}^{n} f(\vec{c}_{ij})\Delta x_i \Delta y_i$$

If this limit exists then we say that f is **Riemann integrable** on R

Theorem 165: If f is continuous on the closed rectangle R, then $\int\int_R f dA$ exists.

Theorem 166: If F is bounded on R and if the set of dicontinuities of f on R has zero area, then $\int\int_R f dA$ exists.

Aside: The set of discontinuitues must have two-dimensional Lebesque measure zero, or in other words $m^2(D) = 0$

Theorem 167 (Fubini's - First Type): Let f be bounded on $R = [a, b] \times [c, d]$ and assune that the set S of discontinuities of f on R has zero area. If every line parallel to the coordinate axes meets S at at most finitely many points, then

$$\int\int_R f dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

Example 168: Say $R := [-2, 2] \times [-1, 3]$ and $f(x, y) = x$. Thus by Fubini's Theorem

$$\begin{aligned} \int\int_R x dA &= \int_{-1}^3 \int_{-2}^2 x dx dy \\ &= \int_{-1}^3 \frac{1}{2} x^2 ||_{-1}^2 dy \\ &= \int_{-1}^3 0 dy \\ &= 0 \end{aligned}$$

Propostion 169: Suppose that f and g are integrable on the closed rectangle R. Then:

1. $f \pm g$ is integrable on R and $\int\int_R f \pm g dA = \int\int_R f dA \pm \int\int_R g dA$
2. αf is integrable $\forall \alpha \mathbb{R}$, $\int\int_R \alpha f dA = \alpha \int\int_R f dA$
3. If $f(x, y) \leq g(x, y) \forall (x, y) \in \mathbb{R}^2$, then $\int\int_R f dA \leq \int\int_R g dA$
4. $|f|$ is also integrable on R and
$$\left| \int\int_R f dA \right| \leq |f| dA$$

Definition 170: We say that D is an **elementary region** in the plane if it can be described as a subset of $\mathbb{R}^2$ of one of the following three types:

Type I (x-simple): $D = \{(x, y) : \gamma(x) \leq y \leq \delta(x), a \leq x \leq b\}$, where $\gamma, \delta : [a, b] \to \mathbb{R}$ are continuous.

Type II (y-simple): $D = \{(x, y) : c \leq y \leq d, \alpha(y) \leq x \leq \beta(y)\}$

Type III: neither Type I nor Type II

Definition 171: If $f(x, y)$ is a function, then the **extension** of f, denoted f^{ext}, is

$$f(x, y)\chi_D = f^{\text{ext}}(x, y) := \left\{ \begin{array}{ccc} f(x, y) & , & (x, y \in D) \\ 0 & , & (x, y) \notin D \end{array} \right\}$$

Remark: In general, f^{ext} is not continuous. However, if f is continuous on $D = \text{ int } D$, then f^{ext} has its discontinuities on ∂D, which has zero area.

Definition 172: Under the previous assumptions and notation, if R is any rectangle such that $R \supseteq D$, then

$$\int\int_D f dA := \int\int_R f^{\text{ext}} dA$$

Theorem 173 (Fubini's Theorem II): Let D bean elementary region on $\mathbb{R}^2$ and f is a continuous function on D. Then:

1. If D is Type I, then
$$\int\int_D f dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x,y) dy dx$$

2. If D is Type II, then
$$\int\int_D f dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x,y) dx dy$$

Proof (Part (1.) only): Let $D := \{(x,y) : \gamma(x) \leq y \leq \delta(x), a \leq x \leq b\}$. By Definition 172, $\int\int_D f dA = \int\int_R f^{\text{ext}} dA$, where R is any rectangle such that $R \supseteq D$. To fix notation, consider the following. $R = [a', b'] \times [c', d']$ where $a' \leq a \leq b \leq b'$ and $c' \leq \gamma(x) \leq \delta(x) \leq d', \forall x \in [a,b]$. Also $R = R_1 \cup R_2 \cup R_3$. Since $f^{\text{ext}} = 0$ outside of R_2, it follows that

$$\int\int_R f^{\text{ext}} dA = \int\int_{R_2} f^{\text{ext}} dA = \int_a^b \int_{c'}^{d'} f^{\text{ext}}(x,y) dy dx$$

Observe that for each fixed x,

$$\int_{c'}^{d'} f^{\text{ext}}(x,y) dy dx = \int_{\gamma(x)}^{\delta(x)} f^{\text{ext}}(x,y) dy$$

So

$$\begin{aligned} \int\int_R f^{\text{ext}} dA &= \int_a^b \int_{c'}^{d'} f^{\text{ext}}(x,y) dy dx \\ &= \int_a^b \int_{\gamma(x)}^{\delta(x)} f^{\text{ext}}(x,y) dy dx \end{aligned}$$

whence

$$\int_D f dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f^{\text{ext}}(x,y) dy dx$$

as desired.

Example 174: Find $\int\int_D ydA$, where D is the area in the plane bounded by $x = y^2$, $x = 0$, and $y = 2$.

$$\begin{aligned}
\int\int_D ydA &= \int_0^4 \int_{\sqrt{x}}^2 ydydx \\
&= \int_0^4 \left[\frac{1}{2}y^2\right]_{\sqrt{x}}^2 dx \\
&= \int_0^4 2 - \frac{1}{2}xdx \\
&= \left[2x - \frac{1}{4}x^2\right]_0^4 \\
&= 4 \\
&= \int_0^2 \int_0^{y^2} ydxdy \\
&= \int_0^2 [yx]_0^{y^2} dy \\
&= \int_0^2 y^3dy \\
&= \left[\frac{1}{4}y^4|\right] 0^2 \\
&= 4
\end{aligned}$$

Example 175:

$$\int_0^{\frac{\pi}{2}} \int_0^{\cos x} \sin xdydx$$

$$\int_0^1 \int_0^{\arccos(y)} \sin xdxdy$$

Example 176:

$$\int_0^3 \int_0^{9-x^2} \frac{xe^{3y}}{9-y}dydx$$

$$\int_0^9 \int_0^{\sqrt{9-y}} \frac{xe^{3y}}{9-y}dxdy$$

$$\int_0^9 \left[frac\frac{1}{2}x^2e^{3y}9-y\right]_0^{\sqrt{9-y}} dy$$

$$\frac{1}{2}\int_0^9 (9-y)\frac{e^{3y}}{9-y}dy$$

$$\frac{1}{2}\int_0^9 e^{3y}dy$$

Definition 177: A **closed box** in $\mathbb{R}^3$ is defined to be the set B, where $B := \{(x, y, z) \in \mathbb{R}^3 : a \leq x \leq b, c \leq y \leq d, p \leq z \leq q\}$ or $B := [a, b] \times [c, d] \times [p, q]$

Definition 178: A **partition** of B, a box in $\mathbb{R}^3$, of order n consists of three collections of **partition points** that divide B into a union of n^3 subboxes.

Definition 179: Let f be any function $\ni f : [a, b] \times [c, d] \times [q, p] \subseteq D \to \mathbb{R}$. Let $\vec{c}_{ijk}$ be any point in the ijk^{th} subbox $B_{ijk} := [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$. Then

$$S := \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f(\vec{c}_{ijk}) \Delta V_{ijk}$$

where $\Delta V_{ijk} := \Delta x_i, \Delta y_j, \Delta z_k$, is called a **Riemann sum** of f corresponding to the partition. Also, **triple integral**.

Example 189:

$$T(u, v) := (u + 1, v + 2)$$
$$S(u, v) := (2u, 3v)$$

Now look at D defined to be the unit square centered at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and transform it with T and S and draw a pretty picture. Or just imagine it, it's that easy.

Remark: We are interested in linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ having the form

$$T(u, v) = (au + bv, cu + dv) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Proposition 190: Let $A := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where det $A \neq 0$. If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(u, v) = A \begin{bmatrix} u \\ v \end{bmatrix}$, then T is bijective, maps parallelograms to parallelograms, and the vertices of parallelograms to the vertices of parallelograms. Moreover, if D^* is a parallelogram in the uv plane that is mapped onto the parallelogram $D := T(D^*)$ in the xy plane, then

$$\text{Area of } D = |\det A|(\text{Area of } D^*)$$

Example 191:

$$T(u, v) = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{aligned} T(0,0) &= (0,0) \\ T(1,3) &= \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = (11,2) \\ T(-1,2) &= (4,3) \\ T(0,5) &= (15,5) \end{aligned}$$

Example 192:

$$T(u,v) = (u, uv)$$

T is niether linear nor injective. Draw your picture if you must but this one is pretty obvious that it maps the unit square into a triangle.

Recall: Change of variables in Calc I (u-substitution)

$$\int_0^2 2x\cos(x^2)dx$$

$$u = x^2$$

$$du = 2xdx$$

$$\int_0^4 \cos(u)du$$

Definition 193: The **Jacobian** of the transformation T, denoted $\frac{\partial(x,y)}{\partial(u,v)}$, is the determinant of the matrix $DT(u,v)$. That is,

$$J := \frac{\partial(x,y)}{\partial(u,v)} = \det\, DT(u,v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Definition 194: Given any differentiable coordinate transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, the Jacobian of T is

$$\frac{\partial(x_1, ..., x_n)}{\partial(u_1, ..., u_n)} = \det\, DT$$

Theorem 195 (Change of Variables Theorem): Let D and D^* be elementary regions in the xy- and uv-planes, respectively. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a coordinate transformation that maps D^* onto D in a one-to-one fashion. If $f : D \to \mathbb{R}$ is any integrable function and we use the transformation T ti make the substitution $x = x(u,v)$, $y = y(u,v)$, then

$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

Examples 196-199: *Some good examples for coordinate transformations.*

Remark (Average Value): The average value of $f(x_1, ..., x_n)$ is

$$\frac{1}{\text{Vol } \Omega} \int_\Omega f d\vec{x}$$

Theorem 200: Let $f[0,1] \to \mathbb{R}$ be Lebesgue integrable. Let $f^+ := \max\{0, f(x)\}$. Then

$$\lim_{n\to\infty} \frac{1}{n} \int_{[0,1]} \ln(1 + e^{nf(x)}) dm(x) = \int_{[0,1]} f^+(x) dm(x)$$

Platform $9\frac{3}{4}$: Let $x : [a, b] \to \mathbb{R}^3$ be a C^1 path. Let $f : X \subseteq \mathbb{R}^3 \to \mathbb{R}$ be continuous such that $X \subseteq x([a, b])$. Let $a = t_0 < t_1 < ... < t_k < ... < t_n = b$ be a partition of $[a, b]$, and let t_k^* be any point in the k^{th} subinterval $[t_{k-1}, t_k]$. Consider the sum

$$\sum_{k=1}^{n} f(x(t_k^*))\Delta s_k$$

where

$$\Delta s_k := \int_{t_{k-1}}^{t_k} ||x'(t)||dt$$

which is the length of the k^{th} segment. Now putting all the segmants sogether we get

$$\lim_{\max\{\Delta s_k\}\to 0} \sum_{k=1}^{n} f(x(t_k^*))\Delta s_k$$

$$\lim_{\max\{\Delta t_k\}\to 0} \sum_{k=1}^{n} f(x(t_k^*))\Delta s_k$$

Note by the Mean Value Theorem for integrals that $\exists t_k^{**} \in [t_(k-1), t_k] \ni$

$$\begin{aligned}\Delta s_k &= \int_{t_{k-1}}^{t_k} ||x'(t)||dt \\ &= (t_k - t_{k-1})||x'(t_k^{**})|| \\ &= ||x'(t_k^{**})||\Delta t_k\end{aligned}$$

Now let $t_k^* = t_k^{**}$. Thus we get

$$\lim_{\max\{\Delta t_k\}\to 0} \sum_{k=1}^{k} f(x(t_k^{**}))||x'(t_k^{**})||\Delta t_k$$

$$\int_a^b f(x(t))||x'(t)||dt$$

Definition 201: *Definition of a line integral.*

Example 202: *A nice example for line integrals.*

Definition 203: The **vector line integral** of $\vec{F}$ along $\vec{x} : [a,b] \to \mathbb{R}^n$, denoted $\int_{\vec{x}} \vec{F} \cdot d\vec{s}$, is

$$\int_{\vec{x}} \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)dt$$

Example 204:

$$\vec{F}(t) = (y+2)\vec{i} + x\vec{j}$$

$$\vec{x}(t) = (\sin t, -\cos t), 0 \le t \le \frac{\pi}{2}$$

$$\vec{F}(\vec{x}(t)) = \vec{F}(\sin t, -\cos t) = (-\cos t + 2)\vec{i} + (\sin t)\vec{j}$$

$$\vec{x}'(t) = (\cos t, \sin t)$$

$$\vec{F} \cdot d\vec{s} = -\cos^2 t + 2\cos t + \sin^2 t = 2\cos t - \cos 2t$$

$$\begin{aligned}\int_{\vec{x}} \vec{F} \cdot d\vec{s} &= \int_0^{\frac{\pi}{2}} 2\cos t - \cos 2t dt \\ &= [2\sin t - \frac{1}{2}\sin 2t]_0^{\frac{\pi}{2}} \\ &= 2 - 0 \\ &= 2\end{aligned}$$

Theorem 215 (Circulation-Curl Form of Green's Theorem): If D is a region to which Green's THeorem applies and $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ is a vector field of class C^1 on D, then orienting ∂D appropriately,

$$\oint_{\partial D} \vec{F} \cdot \vec{T} ds = \oint_{\partial D} \vec{F} \cdot d\vec{s} = \int\int_D (\nabla \times \vec{F}) \cdot \vec{k} dA$$

Theorem 216 (Divergence Theorem in $\mathbb{R}^2$ or Flux-Divergence Form of Green's Theorem): If D is a region to which Green's Theorem applies, $\vec{n}$ is the unit outward normal vector to D, and $\vec{F}(x,y) = M(x,y)\vec{i} + N(x,y)\vec{j}$ is a vector field of class C^1 on D, then

$$\oint_{\partial D} \vec{F} \cdot \vec{n} ds = \int\int_D \nabla \cdot \vec{F} dA$$

Proof: Parametrize ∂D $\vec{x}(t) = (x(t), y(t))$. Note that $\vec{n}$ at the point (x,y) is given by

$$\vec{n} = \frac{y'(t)\vec{i} - x'(t)\vec{j}}{\sqrt{(x'(t))^2 + (y'(t))^2}}$$

Therefore

$$\begin{aligned}
\oint_{\partial D} \vec{F}\cdot\vec{n}ds &= \int_a^b (\vec{F}(\vec{x}(t))\cdot\vec{n}(t))||\vec{x}'(t)||dt \\
&= \int_a^b (M(x(t),y(t))\vec{i}+N(x(t),y(t))\vec{j})\cdot\frac{y'(t)\vec{i}-x'(t)\vec{j}}{\sqrt{(x'(t))^2+(y'(t))^2}}||\vec{x}'(t)||dt \\
&= \int_a^b M(x(t),y(t))y'(t)-N(x(t),y(t))x'(t)dt \\
&= \oint_{\partial D} Mdy - Ndx \\
&= \int\int_D \frac{\partial(M)}{\partial x}-\frac{\partial(-N)}{\partial y}dA \\
&= \int\int_D M_x+N_y dA \\
&= \int\int_D \nabla\cdot\vec{F}dA
\end{aligned}$$

Definition 217: A continuous vector field $\vec{F}$ has **path-independent line integrals** if

$$\int_{c_1}\vec{F}\cdot d\vec{s} = \int_{c_2}\vec{F}\cdot d\vec{s}$$

for any two simple, piecewise C^1 oriented curves lying in the domain of $\vec{F}$ and having the same terminal and initial points.

Theorem 218: Let $\vec{F}$ be a continuous vector field. Then $\vec{F}$ has path-independent line integrals if and only if

$$\oint_c \vec{F}\cdot d\vec{s} = 0$$

for each piecewise C^1 simple closed curve c in the domain of $\vec{F}$.

Assume that $\vec{F}$ is continuous and that $\vec{F}=\nabla f$ for some scalar function f of class C^1. Thus $\vec{F}$ is **conservative** or **gradient**. (Recall that f is called a **potential function** for $\vec{F}$.) Then for each path $\vec{x}$ from $\vec{A}=\vec{x}(a)$ to $\vec{B}=\vec{x}(b)$:

$$\begin{aligned}
\int_{\vec{x}}\nabla f\cdot d\vec{s} &= \int_{\vec{x}}\nabla f\cdot d\vec{s} \\
&= \int_a^b \nabla f(\vec{x}(t))\cdot\vec{x}'(t) \\
&= \int_a^b \frac{d}{dt}(f(\vec{x}(t)))dt \\
&= f(\vec{x}(b)) - f(\vec{x}(a)) \\
&= f(\vec{B}) - f(\vec{A})
\end{aligned}$$

Definition 219: A region $R \subseteq \mathbb{R}^n$ is **connected** if any two points in R can be joined by a path whose image lies in R.

Theorem 220: Let $\vec{F}$ be defined and continuous on a connected, open region R of $\mathbb{R}^n$. Then, $\vec{F} = \nabla f$, where f is C^1 on R, if and only if $\vec{F}$ has path-independent line integrals over curves in R. Moreover, if c is any piecewise C^1 oriented curve in R with initial point $\vec{A}$ and terminal point $\vec{B}$, then

$$\int_c \vec{F} \cdot d\vec{s} = f(\vec{B}) - f(\vec{A})$$

Example 221: Let

$$\vec{F} := 2xy\vec{i} + (x^2 + z^2)\vec{j} + 2yz\vec{k}$$

Note that

$$\nabla(x^2 y + yz^2) = \vec{F}$$

So $\vec{F}$ is conservative. Now let

$$\vec{x}(t) = (t^2, t^3, t^5), 0 \leq t \leq 1$$

where $\vec{x}$ is a parametrization of curve c. Therefore

$$\begin{aligned}\int_{\vec{x}} \vec{F} \cdot d\vec{s} &= f(\vec{x}(1)) - f(\vec{x}(0)) \\ &= f(1,1,1) - f(0,0,0) \\ &= 2\end{aligned}$$

Definition 222: A region R in $\mathbb{R}^2$ or $\mathbb{R}^3$ is **simply connected** if it consists of a single connected piece and if every simple closed curve c in R can be continuously shrunk to a point while remaining in R throughout the transformation.

Definition 223: Let $\vec{F}$ be a vector field of class C^1 where the domain of $\vec{F}$ is a simply connected region R in either $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $\vec{F} = \nabla f$ for some C^2 scalar function f on R if and only if

$$\nabla \times \vec{F} = \vec{0}, \forall \vec{x} \in R$$

Example 224:

$$\vec{F}(x, y, z) = (e^x \sin y - yz)\vec{i} + (e^x \cos y - xz)\vec{j} + (z - xy)\vec{k}$$

Note that $\nabla \times \vec{F} \equiv \vec{0}$. Now by Theorem 223, $\exists f \in C^2(\mathbb{R}^3) \ni \vec{F} = \nabla f$. Thus

$$\begin{aligned}\frac{\partial f}{\partial x} &= e^x \sin y - yz \\ \frac{\partial f}{\partial y} &= e^x \cos y - xz \\ \frac{\partial f}{\partial z} &= z - xy\end{aligned}$$

Now we use all that to solve for f. Based on f_x we start with

$$f(x,y,z) = e^x \sin y - xyz + \psi(y,z)$$

where ψ is an arbitrary function of y and z. Now consider f_y as above and as computed from our tentative f. We notice that since based on our f, $f_y = e^x \cos y - xz + \psi_y(y,z)$. But since from above we have $f_y = e^x \cos y - xz$, $\psi_y(y,z) \equiv \vec{0}$. Thus $\psi(y,z) = \phi(z)$, where ϕ is an arbitrary function of z. Thus

$$f(x,y,z) = e^x \sin y - xyz + \phi(z)$$

But now looking at f_z from above we see that $\phi'(z) = z$ so $\phi(z) = \frac{1}{2}z^2 + C$ and

$$f(x,y,z) = e^x \sin y - xyz + \frac{1}{2}z^2 + C$$

Hogsmeade Station (The Lebesgue Integral): The Riemann integral sucks. Let's make a better one. And let's make it use measure theory somehow. First, let's look at some properties of Riemann integrals.
Notation (Extended Real Numbers):

$$\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$$

Definition 225: A finite sequence of points $P := \{t_0, t_1, ..., t_n\}$ is called a **partition** of $[a,b]$ if $a = t_0 < t_1 < ... < t_n = b$. The **norm** of P is $||P|| := \max_{1 \leq j \leq n}(t_j - t_{j-1})$.
Definition 226: With E a set, put

$$B(E) := \{f : E \to \mathbb{R} : f \text{ is bounded}\}$$

Definition 227: Given $f \in B([a,b])$, the **upper sum** of f with respect to partition P is

$$S_P f := \sum_{j=1}^{n} M_j \cdot (t_j, t_{j-1})$$

where $M_j := \max_{x \in [t_j, t_{j-1}]} f(x)$. Similarly, the **lower sum** of f with respect to P is

$$s_P f := \sum_{j=1}^{n} m_j \cdot (t_j, t_{j-1})$$

where $m_j := \min_{x \in [t_j, t_{j-1}]} f(x)$.
Definition 228: Given $f \in B([a,b])$ the **upper Darboux integral** of f is

$$\overline{I}_a^b(f) = \overline{\int_a^b} f dx := \inf\{S_P f : P \text{ is a partition of } [a,b]\}$$

and the **lower Darboux integral** of f is

$$\underline{I}_a^b(f) = \underline{\int_a^b} f dx := \sup\{s_P f : P \text{ is a partition of } [a, b]\}$$

Platform What's an Infimum?: Say $S := (0, 1]$. $\inf S = 0$ because 0 is a greatest lower bound for S. That is, it is a lower bound and there are no points in S less than it, but also you cannot pick a greater lower bound that is still lower than all the elements in S. It also equals the minimum of the closure of the set since $\overline{S} = [0, 1]$. The supremum, or sup, is the opposite of this. Say you have a set $T := [0, 1)$. The least upper bound of T is 1 for the same reasons as above. Thus $\sup T = 1$.

Proposition 229: $\overline{I}_a^b(f)$ and $\underline{I}_a^b(f)$ always exist and

$$\underline{I}_a^b(f) \le \overline{I}_a^b(f)$$

Definition 230: Let $f \in B([a, b])$ be given. If

$$\underline{I}_a^b(f) = \overline{I}_a^b(f)$$

then f is Reimann integrable, denoted $f \in R([a, b])$, and we write

$$\int_a^b f dx$$

for this common value.

Remark: To show that $f \in R([a, b])$, is suffices to show that

$$\overline{I}_a^b(f) \le \underline{I}_a^b(f)$$

Example 231: Given

$$f(x) \equiv c$$

and

$$f : [a, b] \to \{c\}$$

prove that $f \in R([a, b])$.

Proof: Let P be an arbitrary partition of $[a, b]$. Computing the upper and lower sums we have

$$S_P f = \sum_{j=1}^{n} c(t_j - t_{j-1})$$

$$s_P f = \sum_{j=1}^{n} c(t_j - t_{j-1})$$

Therefore

$$\overline{\int_a^b} f dx = c(b-a) = \underline{\int_a^b} f dx$$

And therefore

$$\int_a^b f dx = c(b-a)$$

And therefore

$$f \in R([a,b])$$

as desired.

Example 232 (Dirichlet Function): Let $f \in B([0,1])$ be given by

$$\chi_{\mathbb{Q}\cap[0,1]} := f(x) = \left\{ \begin{array}{ccc} 0 & , & x \in [0,1]\backslash\mathbb{Q} \\ 1 & , & x \in [0,1]\cap\mathbb{Q} \end{array} \right\}$$

Idea: Show that $\exists \alpha > 0 \ni S_P f - s_P f \geq \alpha$ for all partitions P.

Proof: Let $P := \{t_0, t_1, ..., t_n\}$ be a partition of $[0,1]$. By the completeness of the real numbers we have that for each $j = 1, 2, ..., n$ there is a rational and an irrational in $[t_{j-1}, t_j]$, for $t_{j-1} < t_j$. Hence $M_j = 1$ and $m_j = 0, \forall j$. Therefore

$$\begin{aligned} S_P f - s_P f &= \sum_{j=1}^{n} (M_j - m_j) \cdot (t_j - t_{j-1}) \\ &= \sum_{j=1}^{n} (t_j - t_{j-1}) \\ &= 1 \end{aligned}$$

And so picking any $\alpha \ni 0 < \alpha \leq 1$, say, $\alpha = 1$, we have

$$\overline{I}_a^b f - \underline{I}_a^b f = 1 \neq 0$$

so

$$f \notin R([a,b])$$

as desired.

What's Wrong with Riemann?:

1. Not all bounded functions are Reimann integrable.
2. Unbounded functions must be dealt with differently than bounded functions (i.e., with a limiting argument).

3. The Riemann integral behaves *very* badly with respect to limits.

Theorem 233 (Arzela's Theorem): Let $\{f_n\}_{n=1}^{\infty} \in R([a,b])$ and $\lim_{n\to\infty} f_n(x) = f(x) \in R([a,b])$ be given. Suppose that there is a $g \in R([a,b])$ such that $|f_n(x)| < g(x)$ for all $x \in [a,b]$ and each $n \in \mathbb{N}$. If $f_n(x) \to f(x), \forall x \in [a,b]$, then

$$\begin{aligned}\lim_{n\to\infty} \int_a^b f_n(x)dx &= \int_a^b \lim_{n\to\infty} f_n(x)dx \\ &= \int_a^b f(x)dx\end{aligned}$$

Remark: This is weak because you have to satisfy a ton of hypotheses. We need a better integral for this.
Example 234: Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rationals in $[0,1]$. Put $f_n := \chi_{\cup_{j=1}^{n}\{r_j\}}$. So $\lim_{n\to\infty} f_n = \chi_{\mathbb{Q}\cap[0,1]}$. But

$$\int_0^1 f_n dx = \int_0^1 \chi_{\cup_{j=1}^{n}\{r_j\}} dx = 0, \forall n \in \mathbb{N}$$

and

$$\int_0^1 dx \text{ DNE}$$

So in other words

$$0 = \lim_{n\to\infty} \int_0^1 f_n dx \neq \int_0^1 \lim_{n\to\infty} f_n dx$$

What's Right with Riemann?

1. $R([a,b])$ is a vector space.

2. $\int_a^b f(x)dx \geq 0$ where $f \in R([a,b])$ and $f(x) \geq 0, \forall x \in [a,b]$.

3. The **functional** $f \longmapsto \int_a^b f(x)dx$ is linear (i.e., $\int_a^b f dx$ is a linear transformation).

4. Something like Arzela's Theorem.

5. Fundamental Theorem of Calculus.

Platform $9\frac{3}{4}$: Measure Theory Is really cool.
Definition 235: A collection $\mathcal{T}$ of subsets of X is called a **topology** on X if:

1. $\phi \in \mathcal{T}$, $X \int \mathcal{T}$

2. If $\{U_i\}_{i=1}^{m} \subseteq \mathcal{T}$, then $\cap_{i=1}^{m} U_i \in \mathcal{T}$. (Closure under finite intersection.)

3. If $\{U_\alpha\}_{\alpha\in A} \subseteq \mathcal{T}$, where A is any set, then $\cup_{\alpha\in A} U_\alpha \in \mathcal{T}$. (Closure under arbitrary unions).

Example 236:

1. Given X, $\mathcal{T} := \{\phi, X\}$ is the **trivial topology**.

2. Given X, $\mathcal{T} := P(X)$ is the **discrete topology**.

Definition 237: The elements of $\mathcal{T}$, where $(X, \mathcal{T})$ is a **topological space**, are called **open sets**. The complements of members of $\mathcal{T}$ are called **closed sets**.
Definition 238: A **σ-algebra** is a collection $\mathcal{M}$ of subsets of a set X such that:

1. $\phi \in \mathcal{M}$, $X \in \mathcal{M}$

2. If $E \in \mathcal{M}$, then $E^C \in \mathcal{M}$. (Closure under complements.)

3. If $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{M}$, then $\cup_{i=1}^{\infty} E_i \in \mathcal{M}$. (Closure under countable unions.)

Example 236:

1. Given $X \neq \phi$, $\mathcal{M} := \{\phi, X\}$ is the **trivial σ-algebra**.

2. Given $X \neq \phi$, $\mathcal{M} := P(X)$ is the **discrete σ-algebra**.

3. $X = \mathbb{R}$, $\mathcal{M} := \{\mathbb{R}, \phi, \{0\}, \mathbb{R}\backslash\{0\}\}$

4. $x = \mathbb{R}$, $\mathcal{M} := \{\mathbb{R}, \phi, \{0\}, \{1\}, \{0,1\}, \mathbb{R}\backslash\{0\}, \mathbb{R}\backslash\{1\}, \mathbb{R}\backslash\{0,1\}\}$

5. $X \neq \phi$, $\mathcal{M} := \{E \subseteq X : E \text{ or } E^C \text{ is countable}\}$ Yes, this is a σ-algebra, don't ask me why.

Definition 240: Let $X, \mathcal{M}$ be a measure space, where $\mathcal{M}$ is a σ-algebra. Then μ is called a **measure** on $mathcalM$ if $\mu : \mathcal{M} \to [0, +\infty]$ and

1. $\mu(\phi) = 0$

2. If $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ are mutually disjunct, then $\mu(\cup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j)$. (Countable additivity.)

Example 241: Suppose $X \neq \phi$. Define $\mu : P(X) \to [0, +\infty]$ by

$$\mu(E) := \left\{ \begin{array}{rcc} \text{number of elements of } E & , & E \text{ is finite} \\ +\infty & , & \text{otherwise} \end{array} \right\}$$