Notes on a Series of Lectures on Differential Equations and Linear Algebra

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Spring 2010

Defintion 1: A **differential equation (DE)** is an equation involving an unknown function and certain of its derivatives on some interval.

Remark: We'll initially be interested in

$$y' = f(t, y)$$

Definition 2: The solution to a DE is a function $y(t)$ satisfying the equation on some set I.

Definition 3: An ordinary differential equation (ODE) is called linear if it has the following form

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + ... + a_1(t)y'(t) + a_0(t)y(t) = f(t)$$

Remark: For $n = 1$,

$$a_1(t)y'(t) + a_0(t)y(t) = f(t)$$

and for $n = 2$

$$a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = f(t)$$

Definition 4: The **order** of a DE is equal to the highest derivative present.

Example 5: *We find the slope field for* $y' = t - y$.

Definition 6: An ODE of the form

$$\frac{dy}{dt} = f(y)$$

is called **autonomous**.

Example 7:

$$y' = y\left(1 - \frac{y}{L}\right)$$

$$y' = 0 \Rightarrow y\left(1 - \frac{y}{L}\right) = 0$$

Therfore the equilibrium solutions are $y \equiv 0$ and $y \equiv L$. Now draw a phase line and find that $y \equiv L$ is stable (a sink) while $y \equiv 0$ is unstable (a source).

Definition 8: A first-order DE is called **seperable** if it has the form

$$y' = f(t)g(y)$$

Theorem 9: Given a seperable DE, a solution is given by

$$\int \frac{1}{g(y)}dy = \int f(t)dt$$

Example 10:

$$y' = ty, y(0) = 1$$

$$\int \frac{1}{y} dy = \int t dt$$
$$\ln|y| = \frac{1}{2}t^2 + C$$
$$y(t) = Ae^{\frac{1}{2}t^2}, A := \pm e^C$$

Now plug in the initial condition and we have

$$y(t) = e^{\frac{1}{2}t^2}$$

Platform $9\frac{3}{4}$ (Euler's Method): Given $y' = f(t, y), y(t_0) = y_0$, the slope of a solution y at point (t_0, y_0) is simply $f(t_0, y_0)$. So, pick a step size h, assume the solution is affine or pretty close, and we get that

$$t_n = t_{n-1} + h$$

and

$$y_n = y_{n-1} + hf(t_{n-1}, y_{n-1})$$

Platform $9\frac{3}{4}$ Again (That Was Quick) (Existance and Uniqueness): Let's take a look at a DE with $f(t, y)$ continuous. Perhaps this will guarantee uniqueness.

Example 11:

$$y' = y^{\frac{2}{3}}, y(0) = 0$$

$$\int \frac{1}{y^{\frac{2}{3}}} dy = \int 1 dt$$
$$3y^{\frac{1}{3}} = t + C$$
$$y_1(t) = \frac{1}{27}t^3$$

But wait!

$$y_2(t) \equiv 0$$

is also a solution (the trivial solution). Moreover,

$$y_\xi(t) = \left\{ \begin{array}{rcl} 0 & , & t \leq \xi \\ \frac{1}{27}(t-\xi)^3 & , & t > \xi \end{array} \right\}$$

since this is a shift of the right side of y_1 ξ units right, with the left side replaced by y_2. Notice that this works $\forall \xi \geq 0, \xi \in \mathbb{R}$. So there are infinitely many solutions.

Well that failed. Looks like we have to look for stronger criteria than continuity or differentiablity for uniqness.

Theorem 12 (Picard-Lindelof): Consider the IVP

$$y' = f(t, y), y(t_0) = y_0$$

and suppose that f is Lipschitz continuous in y - that is, $\exists L > 0 \ni |f(t, y_1) - f(t, y_2)| \leq L|y_2 - y_1|$ (it doesn't have to be Lipschitz in t) - on some rectangle about (t_0, y_0). Then the IVP has a unique solution.
Theorem 13: Consider the IVP

$$y' = f(t, y), y(t_0) = y_0$$

Assume that both f and f_y are continuous on some rectangle about (t_0, y_0). Then there is a unique solution that exists on some subrectangle of the rectangle about (t_0, y_0).
Theorem 14 (Cauchy-Peano): Given the IVP

$$y' = f(t, y), y(t_0) = y_0$$

if f is continuous on some rectangle about (t_0, y_0), then the IVP has a solution.
Platform $9\frac{3}{4}$ (Linear Equations): Why? Because even though nothing interesting in the real world is linear, understanding linear equations will help us understand nonlinear equations.
Definition 15: Let L be an operator mapping from some vector space V to some vector space W - that is, $L : V \to W$. Then L is called **linear** if

1. $L(y + z) = Ly + Lz, \forall y, z \in V$

2. $L(\alpha y) = \alpha Ly, \forall \alpha \in \mathbb{R}$ and $\forall y \in V$

Example 16: Some examples of linear operators:

1. $\frac{dy}{dt} =: Ly$

2. $\int_a^b y dt =: Ly$

3. $y'' + 2y' - 3y =: Ly$

4. $y' + \int_0^t y(\tau) d\tau =: Ly$

To prove that say

$$Ly := y'' + 2y' - 3y$$

is linear we would show that

$$\begin{aligned} L(y_1 + y_2) &= (y_1 + y_2)'' + 2(y_1 + y_2)' - 3(y_1 + y_2) \\ &= (y_1'' + 2y_1' - 3y_1) + (y_2'' + 2y_2' - 3y_2) \\ &= Ly_1 + Ly_2 \end{aligned}$$

and

$$\begin{aligned} L(\alpha y_1) &= (\alpha y_1)'' + 2(\alpha y_1)' - 3(\alpha y_1) \\ &= \alpha[y_1'' + 2y_1' - 3y_1] \\ &= \alpha L y_1 \end{aligned}$$

Theorem 17: Let L be linear. Then if

$$Ly_1 = Ly_2 = 0$$

then

1. $L(y_1 + y_2) = 0$

2. $L(\alpha y_1) = 0, \forall \alpha \in \mathbb{R}$

Proof: The proof is practically trivial and no, I'm not going to write it out.

Theorem 18 (Nonhomogeneous Principle): Let L be linear. Then if y_n solves $Ly = 0$ and y_p solves $Ly = f$ (that is, y_p is any one solution to $Ly = f$), for some f, then $y := y_n + y_p$ solves $Ly = f$. Moreover, every solution to $Ly = f$ has the form $y_n + y_p$.

Example 19:

$$y' + ay = b, \text{ where } a, b \in \mathbb{R} \text{ are fixed}$$

Put $Ly := y' + ay$. Then the DE reads $Ly = b$. Now we solve $Ly = 0$.

$$\begin{aligned} Ly &= 0 \\ y' + ay &= 0 \\ y' &= -ay \\ y_h(t) &= Ae^{-at} \end{aligned}$$

Now find any one solution to $Ly = b$ which can be rewritten $y' + ay = b$. Note that

$$y_p(t) \equiv \frac{b}{a}$$

solves $Ly = b$. Thus the general solution to $y' + ay = b$ is

$$y(t) = y_h + y_p = \frac{b}{a} + Ae^{-at}$$

Example 20:

$$x + y = 5$$

First find $\vec{x}_h$ which solves $x + y = 0$.

$$\begin{aligned} x + y &= 0 \\ y &= -x \\ \vec{x}_h &= \begin{bmatrix} c \\ -c \end{bmatrix} = c \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \forall c \in \mathbb{R} \end{aligned}$$

Now find a $\vec{x}_p$, any particular solution to $x + y = 5$. Note that

$$\vec{x}_p = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

works. Thus our solution is

$$\vec{x} = \vec{x}_h + \vec{x}_p = \begin{bmatrix} 0 \\ 5 \end{bmatrix} + c \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \forall c \in \mathbb{R}$$

Platform $9\frac{3}{4}$ (Euler-Lagrange Method): Consider the DE $y' + p(t)y = f(t)$. Put $Ly := y' + p(t)y$. Now solve $Ly = 0$.

$$\begin{aligned} Ly &= 0 \\ y' + p(t)y &= 0 \\ y' &= -p(t)y \\ \int \frac{1}{y} dy &= \int -p(t)dt + C \\ \ln|y| &= \int -p(t)dt + C \\ y_h(t) :&= Ae^{-\int p(t)dt} \end{aligned}$$

where $A := \pm e^C$. Now solve $Ly = f$. Pick as an ansatz

$$y_p(t) := v(t)e^{-\int p(t)dt}$$

Therefore

$$
\begin{aligned}
y_p'(t) &= v'(t)e^{-\int p(t)dt} + v(t)\left[-p(t)e^{-\int p(t)dt}\right] \\
y_p'(t) + p(t)y_p &= v'(t)e^{-\int p(t)dt} - v(t)p(t)e^{-\int p(t)dt} + v(t)p(t)e^{-\int p(t)dt} \\
y_p'(t) + p(t)y_p &= v'(t)e^{-\int p(t)dt} \\
f(t) &= v'(t)e^{-\int p(t)dt} \\
v'(t) &= e^{\int p(t)dt} \\
v(t) &= \int e^{\int p(t)dt} f(t)dt \\
y_p(t) &= \left[\int e^{\int p(t)dt} f(t)dt\right] e^{-\int p(t)dt}
\end{aligned}
$$

Thus our final solution is

$$
y = y_p + y_h = e^{-\int p(t)dt}\left[A + \int e^{\int p(t)dt} f(t)dt\right]
$$

Platform $9\frac{3}{4}$ (Integrating Factor Method): Start with

$$
y' + p(t)y = f(t)
$$

Multiply through by some integrating factor $\mu(t)$ to get

$$
\mu(t)y' + \mu(t)p(t)y = \mu(t)f(t)
$$

The question is if we can find a $\mu(t)$ so that

$$
\mu(t)y' + \mu(t)p(t)y = [\mu(t)y]'
$$

Solving for this μ we get

$$
\begin{aligned}
\mu' y + \mu y' &= \mu y' + \mu p y \\
\mu' y &= \mu p y \\
\frac{\mu'(t)}{\mu(t)} &= p(t) \\
[\ln(\mu(t))]' &= p(t) \\
\ln(\mu(t)) &= \int p(t)dt \\
\mu(t) &= e^{\int p(t)dt}
\end{aligned}
$$

The key facts are this formula for picking μ, the fact from which all this is derived,

$$
[\mu y]' = \mu f
$$

and

$$y = \frac{1}{\mu}\int \mu f dt + \frac{C}{\mu}$$

Example 21:

$$y' + 2y = 0$$

First use the IFM. Put $\mu(t) := e^{\int 2dt} = e^{2t}$. Now

$$\begin{aligned}
[e^{2t}y]' &= e^{2t}\cdot 0 = 0 \\
e^{2t}y &= \int 0dt = 0 + C \\
y(t) &= Ce^{-2t}
\end{aligned}$$

Alternatively, you can use Linear Operator Theory. Clearly

$$y_p \equiv 0$$

On the other hand,

$$y_h = Ce^{-2t}$$

because

$$y' = -2y \Rightarrow y = Ce^{-2t}$$

Therefore

$$y = y_h + y_p = Ce^{-2t}$$

Definition 22: We call $R : C^1(I) \to C(I)$ the **Riccati operator** if R has the form

$$Rz(t) := z'(t) + q(t) + \frac{1}{p(t)}z^2(t)$$

for $t \in I$. (Assume that $p, q \in C(I)$.) Also, the DE

$$z' + q + \frac{1}{p(t)z^2} = 0$$

is called the **Riccati DE**.

Remark: The Riccati DE is used in

1. Oscillation theory (Fite-Winthes, Emden-Fowler)

2. Calculus of variations

Theorem 23 (Factorization Theorem): Assume $x \in C^1(J)$, where $J \subseteq \mathbb{R}$, and $x(t) \neq 0$ on J. Then if

$$z(t) := \frac{p(t)x'(t)}{x(t)}, t \in J$$

then

$$Lx(t) = x(t)Rz(t), \forall t \in J$$

where R is the Riccati operator and L is the operator

$$Lx := (p(t)x'(t))' + q(t)x(t)$$

which is the second order self-adjoint operator.

Proof: Assume that $x \in C^1(J)$, $x(t) \neq 0$, and put

$$z(t) := \frac{p(t)x'(t)}{x(t)}, t \in J$$

which is the so-called Riccati substitution. So:

$$\begin{aligned} xRz &= x\left[z' + q + \frac{z^2}{p}\right] \\ &= x\left[\left(\frac{px'}{x}\right)' + q + \frac{1}{p}\left(\frac{px'}{x}\right)^2\right] \\ &= x\left(\frac{px'}{x}\right)' + qx + \frac{x}{p}\left(\frac{px'}{x}\right)^2 \\ &= x\left[\frac{(px')'x - (px')x'}{x^2}\right] + qx + \frac{x}{p} \cdot \frac{p^2(x')^2}{x^2} \\ &= \frac{x^2(px')'}{x^2} - \frac{(px')x'x}{x^2} + qx + \frac{p^2x(x')^2}{px^2} \\ &= (px')' + qx \\ &= Lx \end{aligned}$$

King's Cross Station (Applications):

Platform $9\frac{3}{4}$ (Exponential Decay/Growth): Malthusian growth, circa 1797, says

$$\frac{dy}{dt} = ry$$

where y is the amount of some quantity and r is the decay rate if $r < 0$ and the growth rate is $r > 0$. To solve this

$$\begin{aligned}\frac{dy}{dt} &= ry \\ \int \frac{1}{y} dy &= \int r dt \\ \ln |y| &= rt + C \\ y(t) &= Ae^{rt}\end{aligned}$$

and after solving for A given initial condition $y(0) = y_0$

$$y(t) = y_0 e^{rt}$$

Platform $9\frac{3}{4}$ (Annuity Problem):

$$\frac{dA}{dt} = rA + a, A(0) = A_0$$

where A is the amount of money, a is the amount deposited per time unit, r is the interest rate, and A_0 is the initial deposit. The units on $\frac{dA}{dt}$ are money per time, the units on A are money, the units on r are per time and the units on a is money per time. Solving this using the IFM, first rewriting it in the proper form

$$\begin{aligned}A' &= rA + a \\ A' - rA &= a\end{aligned}$$

then finding $\mu(t)$

$$\mu(t) = e^{\int -r dt} = e^{-rt}$$

and finally solving for $A(t)$

$$\begin{aligned}[e^{-rt}A]' &= ae^{-rt} \\ e^{-rt}A &= -\frac{a}{r}e^{-rt} + C \\ A(t) &= -\frac{a}{r} + Ce^{rt}\end{aligned}$$

and then go back to solve for C

$$\begin{aligned}A(0) &= A_0 \\ &= C - \frac{a}{r} \\ C &= A_0 + \frac{a}{r}\end{aligned}$$

so

$$\begin{aligned} A(t) &= -\frac{a}{r} + \left(A_0 + \frac{a}{r}\right) e^{rt} \\ &= \frac{a}{r}(e^{rt}) + A_0 e^{rt} \end{aligned}$$

Platform $9\frac{3}{4}$ (Newton's Law of Cooling):

$$\frac{dT}{dt} = k(M_0 - T), k > 0$$

where T is the object's temperature, M_0 is the surrounding temperature, and k is the thermal diffusity constant.

Platform $9\frac{3}{4}$ (Mixing Tanks):

$$\frac{dx}{dt} = r_i c_i - r_o c_o$$

where c_i is the concentration of liquid flowing into the tank, r_i is the rate of flow into the tank, c_o is the concentration of liquid flowing out of the tank, r_o is the rate of flow out of the tank, and x is the amount of substance in the tank. The units on $\frac{dx}{dt}$ are amount per time, the units on c_i and c_o are amount per volume, and the units on r_i and r_o are volume per time.

Example 24a (Logistic Model):

$$y' = ry\left(1 - \frac{y}{L}\right)$$

where L is the carrying capacity, y is the number of individuals, and r is the intrinsic exponential growth rate. We can draw a phase line to see that there is a stable equilibrium at $y = L$ and an unstable equilibrium at $y = 0$.

Example 24b (Logistic Model with Threshold):

$$y' = ry\left(1 - \frac{y}{L}\right)\left(\frac{y}{T} - 1\right)$$

where $T < L$ is the threshold population. Drawing the phase line again, we find a stable equilibrium at $y = L$, an unstable equilibrium at $y = T$. All populations below T crash to 0 and all 0 populations remain there.

Example 25 (Supercritical Pitchfork Bifurcation):

$$\frac{dy}{dt} = \alpha y - y^3, \alpha \in \mathbb{R}$$

For example, for $\alpha = 0$, we get

$$y' = -y^3$$

and so $y = 0$ is a stable equilibrium. Instead of doing this for all α, we can draw a bifurcation diagram. Solving when $\alpha > 0$,

$$\begin{aligned} \alpha y - y^3 &= 0 \\ \alpha - y^2 &= 0 \\ y^2 &= \alpha \\ y &= \pm\sqrt{\alpha} \end{aligned}$$

Drawing the phase line for this, we get that there are stable equilibria at $y = \sqrt{\alpha}$ and $y = -\sqrt{\alpha}$ and an unstable equilibrium at $y = 0$. Now for $\alpha < 0$, there are no $\alpha \in \mathbb{R}$ that satisfy $\alpha y - y^3 = 0$, so there is only a stable equilibrium at $y = 0$.

Platform $9\frac{3}{4}$ (Systems): Consider the general autonomous first-order system

$$\left\{ \begin{array}{lcl} \frac{dx}{dt} = f(x,y) & , & x(t_0) = x_0 \\ \frac{dy}{dt} = g(x,y) & , & y(t_0) = y_0 \end{array} \right\}$$

Basically you plot the solution parametrically surpressing the dependence on time. Now by the parametric derivate rule

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Now, for a point to be an equilibrium, both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ must be 0.

Example 26 (Lotka-Voltera Model):

Ideas:

- Foxes are predators.
- Rabbits are prey.
- In the absence of rabbits, the fox population decreases exponentially.
- In the absense of foxes, the rabbit population increases exponentially.
- We model the interaction between rabbits and foxes by a mass-action term – namely, (number of foxes)×(number of rabbits).

So we have

$$\begin{aligned} \frac{dR}{dt} &= a_R R - c_R RF \\ \frac{dF}{dT} &= -a_F F + c_F RF \end{aligned}$$

where R is the number of rabbits, F is the number of foxes, a_R is the intrinsic growth rate for rabbits, a_F is the intrinsic decay rate for foxes, and c_R and c_F describe how bad an encounter of a rabbit with a fox is for the rabbit and how good it is for the fox, respectively. Note that all the constants are always nonnegative. Plugging in some constants for the sake of the example, namely, we get

$$\frac{dR}{dt} = 2R - RF$$
$$\frac{dF}{dT} = -3F + 2RF$$

Solving this for equlibria we get

$$R(2 - F) = 0$$
$$F(-3 + 2R) = 0$$

To continue, we can find the h-nullclines (lines made of points where the slope is horizontal). So set $\frac{dF}{dt} = 0$ and solve.

$$-3F + 2RF = 0$$
$$F(-3 + 2R) = 0$$

So we have h-nullclines at $F = 0$ and $R = \frac{3}{2}$. Now finding the v-nullclines (lines made of points where the slope is vertical). So set $\frac{dR}{dt} = 0$ and solve.

$$2R - RF = 0$$
$$R(2 - F) = 0$$

So we have v-nullclines at $R = 0$ and $F = 2$. Now pick tests points in each reagion between nullclines to see that some sort of counterclockwise rotation is going on here. However, we don't know if solutions spiral out, spiral in, or cirlce forever. However, the solutions do turn out to be closed loops. So obvioulsy it's the circle of life and we need to watch The Lion King.

Example 27 (Competition Model):

$$\frac{dR}{dt} = R(3 - R - 2S) = R(3 - R) - 2RS$$
$$\frac{dS}{dt} = S(2 - S - R) = S(2 - S) - SR$$

Find the equilibrium:

$$\begin{aligned} R(3-R-2S)=0 &\Longrightarrow R=0 \\ & \quad\;\; R=3-2S \\ S(2-S-R)=0 &\Longrightarrow S=0 \\ & \quad\;\; S=2-S \end{aligned}$$

So the equilibria are (0,0), (0,2), (3,0), and (1,1). The h-nullclines are

$$S=0$$
$$S=-R+2$$

and the v-nullclines are

$$R=0$$
$$S=-\frac{1}{2}R+\frac{3}{2}$$

Platform $9\frac{3}{4}$ (Math Epidemiology): Some possible things to model: susceptibles, infectives, recovered, vector, contageous asymptomatic, quarantine. This model was developed in the 1920s by Kermack and McKendrick.

Example 28 (SI Model):

$$\frac{dS}{dt}=-\alpha SI, S(0)=S_0$$
$$\frac{dI}{dt}=\alpha SI, I(0)=I_0$$

where S is the number of susceptibles, I is the number of infectives, and α is the rate of infectivity. Notice that

$$\frac{dS}{dt}+\frac{dI}{dt}\equiv 0 \Leftrightarrow (S+I)(t)\equiv N$$

Therefore

$$S'(t)=-\alpha S(N-S)$$

and by seperation of variables

$$S(t)=\frac{(N-1)N}{(N-1)+e^{N\alpha t}}$$

Example 29 (SIS model):

$$\frac{dS}{dt}=-\alpha SI+\beta I, S(0)=S_0$$
$$\frac{dI}{dt}=\alpha SI-\beta I, I(0)=I_0$$

where S is the number of susceptibles, I is the number of infectives, α is the rate of infectivity, and β is the rate of recovery. Note that

$$\frac{dS}{dt} + \frac{dI}{dt} \equiv 0 \Leftrightarrow (S+I)(t) \equiv N$$

Therefore

$$\begin{aligned} I'(t) &= \alpha S(N-I) - \beta I \\ &= (\alpha N - \beta)I - \alpha I^2 \\ &= \kappa I - \alpha I^2 \end{aligned}$$

where $\kappa := \alpha N - \beta$. Solving this we get

$$I(t) = \left\{ \begin{array}{ccc} \frac{e^{\kappa t}}{\frac{\alpha}{\kappa}[e^{\kappa t}-1]+\frac{1}{I_0}} & , & \kappa \neq 0 \\ \frac{1}{\alpha t + \frac{1}{I_0}} & , & \kappa = 0 \end{array} \right\}$$

Now consider the following limits:

$$\begin{aligned} \kappa > 0 &\Rightarrow \lim_{t\to 0} I(t) = \frac{\kappa}{\alpha} \\ \kappa = 0 &\Rightarrow \lim_{t\to 0} I(t) = 0 \\ \kappa < 0 &\Rightarrow \lim_{t\to 0} I(t) = 0 \end{aligned}$$

so for $\kappa > 0$, an epidemic occurs, and for $\kappa \leq 0$, no epidemic occurs.

Example 30 (Anderson et al., 1986, HIV→AIDS):

$$\begin{aligned} \frac{dx}{dt} &= -\nu(t)x, x(0) = 1 \\ \frac{dy}{dt} &= \nu(t)x, y(0) = 0 \end{aligned}$$

where x is the proportion of the population in the HIV state and y is the proportion of the population in the AIDS state. $\nu(t)$ is a given function, which was statistically approximated by $\nu(t) = 0.237t$. Note that $x + y = 1$. Solving for x, we get

$$\frac{dx}{dt} + \nu(t)x(t) = 0$$

$$\begin{aligned} \ln x(t) &= \int -\nu(t)dt + C \\ \ln x(t) &= -\frac{0.237}{2}t^2 + C \\ x(t) &= e^{-\frac{0.237}{2}t^2} \end{aligned}$$

and so

$$y(t) = 1 - e^{-\frac{0.237}{2}t^2}$$

Platform $9\frac{3}{4}$ (Some Notation):

$$\left.\begin{cases} \frac{dx}{dt} = f(x,y) \quad , \quad x(t_0) = x_0 \\ \frac{dy}{dt} = g(x,y) \quad , \quad y(t_0) = y_0 \end{cases}\right\} \Leftrightarrow \vec{y}' = \vec{f}(t, \vec{y})$$

Definition 31: Consider the ODE $\vec{x}'(t) = \vec{f}(t, \vec{x})$, where $f : T \times \Omega \to \mathbb{R}^n$, where $T \subseteq \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$. Then $\vec{\phi}(t, \vec{x}_0)$ detnotes the collection of all solutions to the DE with initial condition $\vec{x}(0) = \vec{x}_0$. Thus, $\vec{\phi}$ is the **flow** of the DE.

Example 32:

$$y' - 2ty = 8t$$

We will use Euler-Lagrange to solve this. First find y_h:

$$y' - 2ty = 0$$

$$y' = 2ty$$

$$y_h = Ae^{t^2}$$

Now for y_p:

$$v' = 8te^{-t^2}$$

$$v = -4e^{-t^2}$$

Therefore

$$y_p = -4e^{-t^2}e^{t^2} = -4$$

And so

$$y = y_h + y_p = Ae^{t^2} - 4$$

Now to find the flow, set an initial condition $y(0) = y^*$. Then

$$y^* = Ae^{0^2} - 4$$

$$A = y^* + 4$$

And so

$$y(t) = (y^* + 4)e^{t^2} - 4$$

And the flow is written as

$$\phi(t, y) = (y + 4)e^{t^2} - 4$$

From this we can analyze the solutions qualitatively.

Example 33:

$$x_1' = -2x_1 x_2' \qquad\qquad = -3x_2 + 1$$

Note that this can be written in the form $\vec{x}' = A\vec{x} + \vec{f}$; specifically,

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Looking at the first equation, we have

$$x_1 = Ae^{-2t}$$

The second equation can be solved by the integrating factor method.

$$\mu(t) = e^{\int 3dt} = e^{3t}$$

$$(x_2 e^{3t})' = e^{3t} \cdot 1$$
$$x_2 e^{3t} = \frac{1}{3}e^{3t} + C$$
$$x_2 = \frac{1}{3} + Ce^{-3t}$$

Now apply the initial condition $x_1(0) = x_1^0$ and $x_2(0) = x_2^0$.

$$x_1(0) = A = x_1^0$$

$$x_2(0) = \frac{1}{3} + C = x_2^0$$
$$C = x_2^0 - \frac{1}{3}$$

And therefore the flow is

$$\phi(t, x_1, x_2) = \left(x_1 e^{-2t}, \frac{1}{3} + \left(x_2 - \frac{1}{3}\right)e^{-3t}\right)$$

Definition 34: Let $\vec{x}_0 \in \mathbb{R}^n$ and $r > 0$. Let $\vec{x}'(t) = f(\vec{x})$ be an ODE in n unknown functions and in n equations. Suppose that $\vec{x}_0$ is an equilibrium point for $\vec{x}' = f(\vec{x})$ (that is, $f(\vec{x}_0) = \vec{0}\forall t$). Then:

1. $\vec{x}_0$ is **stable** if for each open ball $\beta_r(\vec{x}_0)$ there is another open ball $\beta_s(\vec{x}_0)$, then $\phi(t, \vec{x}) \in \beta_r(\vec{x}_0), \forall t > 0$.

2. If, in addition to (1.), $\exists \beta_p(\vec{x}_0)$ so that for each $\vec{x} \in \beta_p(\vec{x}_0)$, $\phi(t, \vec{x}) \to \vec{x}_0$ as $t \to \infty$, then $\vec{x}_0$ is **asymptotically stable**.

3. If neither (1.) nor (2.) holds, then $\vec{x}_0$ is **unstable**.

Definition 35: A set $S \subset \mathbb{R}^n$ is said to be **positively** or **forward invariant** under the system

$$\vec{x}'(t) = \vec{f}(\vec{x})$$

if for each $\vec{x}_0 \in S$,

$$\phi(t, \vec{x}_0) \in S, \forall t \geq 0$$

Definition 36: The ω**-limit set** of a flow $\phi(t, \vec{x}_0)$ is

$$\omega(\vec{x}_0) := \{\vec{x} \in \mathbb{R}^n : \exists (t_k)_{k=1}^{\infty} \ni t_k \to \infty \Rightarrow \phi(t_k, \vec{x}_0) \to \vec{x}\}$$

Also, the α**-limit set** is

$$\alpha(\vec{x}_0) := \{\vec{z} \in \mathbb{R}^n : \exists (t_k)_{k=1}^{\infty} \ni t_k \to -\infty \Rightarrow \phi(t_k, \vec{x}_0) \to \vec{z}\}$$

Example 37:

$$y' = -5y$$

$$y(t) = Ae^{-5t}$$

$$y(0) = y_0 \Rightarrow y_0 = Ae^0 \Rightarrow A = y_0$$

Therfore the flow is

$$\phi(t, y) = ye^{-5t}$$

Now analyzing the solutions, we see that all solutions tend to zero and therefore

$$\omega(y) = \{0\}, \forall y \in \mathbb{R}$$

Now looking at the reverse direction,

$$\alpha(y) = \left\{ \begin{array}{ccc} \emptyset & , & y \neq 0 \\ \{0\} & , & y = 0 \end{array} \right\}$$

Example 38:

$$x_1' = -x_1$$
$$x_2' = x_2 + x_1^2$$

$$x_1(t) = Ae^{-}t$$

Plugging this into the second equation we get

$$x_2' - x_2 = A^2 e^{-2t}$$

Solving this you eventually get

$$x_2(t) = -\frac{A^2}{3}e^{-2t} + Be^t$$

Now to find the flow consider the initial conditions $x_1(0) = x_1^0$ and $x_2(0) = x_2^0$. So

$$x_1^0 = A$$

and

$$\begin{aligned} x_2^0 &= -\frac{A^2}{3} + B \\ &= -\frac{(x_1^0)^2}{3} + B \end{aligned}$$

$$B = x_2^0 + \frac{(x_1^0)^2}{3}$$

And therefore the flow is

$$\vec{\phi}(t, x_1, x_2) = \left(x_1 e^{-t}, -\frac{x_1^2}{3}e^{-2t} + \left(\frac{x_1^2}{3} + x_2\right)e^t\right)$$

Now notice that

$$\phi(t, x_1, x_2) \to \vec{0} \Leftrightarrow \frac{x_1^2}{3} + x_2 = 0$$

and

$$\frac{x_1^2}{3} + x_2 \neq 0 \Rightarrow ||\vec{\phi}(t, x_1, x_2)||_2 \to \infty$$

So $x_2 = -\frac{x_1^2}{3}$ is called a stable manifold of this system. This holds because the set

$$\mathcal{S} := \left\{\vec{x} \in \mathbb{R}^2 : x_2 = -\frac{x_1^2}{3}\right\}$$

is invariant since

$$\begin{aligned} \vec{\phi}(t, x_1, x_2) &= \left(x_1 e^{-t}, -\frac{x_1^2}{3}e^{-2t} + \left(\frac{x_1^2}{3} - \frac{x_1^2}{3}\right)e^t\right) \\ &= \left(x_1 e^{-t}, -\frac{x_1^2}{3}e^{-2t}\right) \end{aligned}$$

which is in $\mathcal{S}$.

Definition 39: Let $\vec{x}_0$ be an equilibrium point for the system $\vec{x}' = f(\vec{x})$, where $\vec{f} \in C^1(\Omega)$ and $\Omega \in \mathbb{R}^n$. A continuously differentiable function V defined in $\Omega_0 \in \Omega$ with $\vec{x}_0 \in \Omega_0$ is called a **Lyapunov function** for the system $\vec{x}' = f(\vec{x})$ on Ω_0 provided that $V(\vec{x}_0) = 0$, $V(\vec{x}) > 0$ for $\vec{x} \neq \vec{x}_0$, and $\frac{d}{dt}V(\vec{x}) = \nabla V(\vec{x}) \cdot f(\vec{x}) \leq 0, \forall \vec{x} \in \Omega_0$. If we replace $\frac{d}{dt}V(\vec{x}) \leq 0$ with $\frac{d}{dt}V(\vec{x}) < 0$, then V is **strict Lyapunov**.

Theorem 40 (Lyapunov's Direct Method, 1892): If V is Lyapunov for $\vec{x}' = f(\vec{x})$ on an open set Ω containing $\vec{x}_0$, then $\vec{x}_0$ is stable. Furthermore, if V is strict Lyapunov, then $\vec{x}_0$ is asymptotically stable. Finally, if $V > 0, \forall \vec{x} \in \Omega \backslash \{\vec{x}_0\}$, then $\vec{x}_0$ is unstable.

Corollary 41: If V is Lyapunov for $\vec{x}' = f(\vec{x})$ on a bounded open set Ω, then $\forall c > 0$ constant such that $\{\vec{x} \in \Omega : V(\vec{x} \leq c)\}$ is closed, the set $\{\vec{x} \in \Omega : V(\vec{x} \leq c)\}$ is positively invariant.

Example 42:

$$\begin{aligned} x' &= -x - xy^2 \\ y' &= -y + 3x^2 y \end{aligned}$$

Now (0,0) is an equilibrium. Put

$$V(x, y) = \alpha x^2 + \beta y^2$$

Now

$$\begin{aligned} \frac{d}{dt} V(x, y) &= \nabla V \cdot \vec{f} \\ &= (a\alpha x, 2\beta y) \cdot (-x - xy^2, -y + 3x^2 y) \\ &= -2\alpha x^2 - 2\alpha x^2 y^2 - 2\beta y^2 + 6\beta x^2 y^2 \end{aligned}$$

Pick $\alpha = 3$ and $\beta = 1$. Then

$$\dot{V} = -6x^2 - 2y^2 < 0, \forall \vec{x} \in \mathbb{R}^2 \backslash \{(0, 0)\}$$

Since $V(vec0) = 0$ and $V > 0$ on $\mathbb{R}^2 \backslash \{\vec{0}\}$, V is a Lyapunov function and (0,0) is globally asymptotically stable. You can then build a set

$$S := \{\vec{x} \in \mathbb{R}^2 : 3x^2 + y^2 \leq c\}$$

which is positively invariant.

Example 43:

$$\begin{aligned} x' &= -2y + yz \\ y' &= x - xz \\ z' &= xy \end{aligned}$$

Note that (0,0,0) is an equilibrium. Put

$$V(x, y, z) := \alpha x^2 + \beta y^2 + \gamma z^2$$

Now

$$\begin{aligned}\dot{V} &= (2\alpha x, 2\beta y, 2\gamma z) \cdot (-2y + yz, x - xz, xy)\\ &= -4\alpha xy + 2\alpha xyz + 2\beta xy - 2\beta xyz + 2\gamma xyz\\ &= xy(-4\alpha + 2\beta) + xyz(2\alpha - 2\beta + 2\gamma)\end{aligned}$$

So we need a solution to

$$\begin{aligned}-4\alpha + 2\beta &= 0\\ 2\alpha - 2\beta + 2\gamma &= 0\end{aligned}$$

one of which is $\alpha = \gamma = 1, \beta = 2$. Therefore V is a non-strict Lyapunov function.

Theorem 44 (LaSalle's Invariance Theorem, ca. 1964): Let V be Lyapunov for $\vec{x}' = f(\vec{x})$ on a bounded open set $\Omega \subseteq \mathbb{R}^n \ni \vec{x}_0 \in \Omega$, where $\vec{x}_0$ is an equilibrium point. If $c > 0$ is a constant so that $S := \{\vec{x} : V(\vec{x}) \leq c\}$ is a closed set in $\mathbb{R}^n$, and if there exists no $\vec{x} \neq \vec{x}_0$ in S such that $V(\phi(t, \vec{x}))$ is constant for $t \geq 0$, then $\forall \vec{x} \in S$, $\phi(t, \vec{x}) \to \vec{x}_0$ as $t \to \infty$.

Example 45:

$$\begin{aligned}x' &= y\\ y' &= -x - y\end{aligned}$$

$$V(x, y) = x^2 + y^2$$

Example 46:

$$\begin{aligned}x' &= y\\ y' &= -x + y^5 - 2y\end{aligned}$$

$$V(x, y) = x^2 + y^2$$

Platform $9\frac{3}{4}$ (Vector Spaces, Revisited):

Flashback: Definition 1 from Notes on a Series of Lectures on Vector Calculus: Let $\mathbb{V}$ be a set. $\mathbb{V}$ is called a **vector space** provided that $\forall x, y, z \in \mathbb{V}$ and $\alpha, \beta \in \mathbb{R}$:

1. $x + y = y + x$
2. $\alpha(x + y) = \alpha x + \alpha y$
3. $(\alpha + \beta)x = \alpha x + \beta x$
4. $\exists 0 \in \mathbb{V} \ni x + 0$

5. $1 \cdot x = x$
6. $(x + y) + z = x + (y + z)$
7. $\exists \tilde{x} \ni x + \tilde{x} = 0$
8. $\alpha(\beta x) = (\alpha \cdot \beta)x$
9. $x + y \in \mathbb{V}$
10. $\alpha x \in \mathbb{V}$

Example 47: Some examples of vector spaces include

$$\mathbb{R}^n$$

$$C^k(\Omega)$$

$$M_{m\times n}(\mathbb{R})$$

$$\mathbb{P}_n(\mathbb{R})$$

Example 48: Some sets that are not vector spaces:

$$\mathbb{R}^3_+$$

$$\mathbb{P}_E$$

Example 49: Let $\mathbb{S}$ be the solutions to

$$y'' + p(t)y' + q(t)y = 0$$

where $p, q : C(\tau) \to \mathbb{R}$ are given.
Claim: $\mathbb{S}$ is a vector space.
Outline of Proof:
Closure under scalar multiplication: Let $u \in \mathbb{S}$. So, u solves the DE. Let $c \in \mathbb{R}$. Then:

$$\begin{aligned}(cu)'' + p(t)(cu)' + q(t)(cu) &= cu'' + cp(t)u' + cq(t)u \\ &= c[u'' + p(t)u' + q(t)u] \\ &= c \cdot 0 \\ &= 0\end{aligned}$$

Therefore $cu \in \mathbb{S}$.

Closure under vector addition: Let $u, v \in \mathbb{S}$. Then:

$$\begin{aligned}[u+v]'' + p(t)[u+v]' + q(t)[u+v] &= [u'' + p(t)u' + q(t)u] + [v'' + p(t)v' + q(t)v] \\ &= 0 + 0 \\ &= 0\end{aligned}$$

Therefore $u + v \in \mathbb{S}$.

Theorem 50 (Cancellation Law): If $x, y, z \in \mathbb{V}$ a vector space such that $x + z = y + z$, then $x = y$.

Corollary 51: Given a vector space, the zero vector is unique.

Theorem 52: In any vector space $\mathbb{V}$, the following hold:

1. $0x = 0, \forall x \in \mathbb{V}$
2. $(-a)x = -(ax) = a(-x), \forall a \in \mathbb{R}, x \in \mathbb{V}$
3. $a0 = 0, \forall a \in \mathbb{R}$

Definition 53: A subset W of a vector space V over a field F is called a **subspace** of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V.

Remark: Given any V, $\{0\}$ and V are trivial subspaces.

Theorem 54: Let $\mathbb{V}$ be a vector space and $\mathbb{W} \subseteq \mathbb{V}$. Then $\mathbb{W}$ is a subspace if and only if

1. $0 \in \mathbb{W}$
2. $x + y \in \mathbb{W}$ wherever $x, y \in \mathbb{W}$
3. $\alpha x \in \mathbb{W}$ wherever $x \in \mathbb{W}, \alpha \in \mathbb{R}$

Example 55 (Example of a Subspace):

$$V := C([0,1])$$

Prove: That

$$W := \{f \in C([0,1]) : f(0) = f(1) = 0\}$$

is a subspace.

Proof:

1. $0 \in W$

2. $x + y \in W$ whenever $x, y \in W$

$$(x+y)(0) = x(0) + y(0) = 0 + 0 = 0$$
$$(x+y)(1) = x(1) + y(1) = 0 + 0 = 0$$

3. $cx \in W$ whenever $c \in \mathbb{R}$, $x \in W$

$$cx(0) = 0$$
$$cx(1) = 0$$

Nonexample:

$$W := \{(a, 0, b, 1, c) : a, b, c \in \mathbb{R}\} \subseteq \mathbb{R}^5$$

is not a subspace of $\mathbb{R}^5$ since $\vec{0} \notin W$.

Definition 56: Let V be a vector space and $S \subseteq V$ with $S \neq \emptyset$. A vector $v \in V$ is called a **linear combination** of vectors in S if there exist vectors $u_1, ..., u_n \in V$ and scalars $a_1, ..., a_n \in \mathbb{R}$ such that

$$v = \sum_{i=1}^{n} a_i u_i$$

Example 57: Write $(2, 6, 8) \in \mathbb{R}^3$ as a linear combination of

$$u_1 = (1, 2, 1)$$
$$u_2 = (-2, -4, -2)$$
$$u_3 = (0, 2, 3)$$
$$u_4 = (2, 0, -3)$$
$$u_5 = (-3, 8, 16)$$

So

$$(2, 6, 5) = a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5$$

and to solve this, solve the system

$$a_1 - 2a_2 + 2a_4 - 3a_5 = 2$$
$$2a_1 - 4a_2 + 2a_3 + 8a_5 = 6$$
$$a_1 - 2a_2 + 3a_3 - 3a_4 + 16a_5$$

And after some intermediate steps we find that one possible solution is

$$\begin{aligned} a_1 &= -4 \\ a_2 &= 0 \\ a_3 &= 7 \\ a_4 &= 3 \\ a_5 &= 0 \end{aligned}$$

Definition 58: Let $S \neq \emptyset$ satisfy $S \subseteq V$, where V is a vector space. Then the **span** of S, denoted $\text{span}(S)$, is the set consisting of all linear combinations of the vectors in S.
Remark: $\text{span}(\emptyset) := \{0\}$
Example 59:

$$\text{span}(\{(1,0,0),(0,1,0)\}) = xy\text{-plane}$$

Theorem 60: The span of any subset S of a vector space V is a subspace of V. Moreover, any subspace of V containing S must contain the $\text{span}(S)$, or, in other words,

$$V \supseteq S \Rightarrow V \supseteq \text{span}(S)$$

Definition 61: A subset S of a vector space V **generates** (or **spans**) V if $\text{span}(S) = V$. In this case, we say taht the vectors of S generate V.
Example 62:

$$S := \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Prove or disprove that

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \text{span}(S)$$

Does

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = c_1v_1 + c_2v_2 + c_3v_3$$

have a solution?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = c_1v_1 + c_2v_2 + c_3v_3$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 & c_2 + c_3 \\ -c_1 & c_2 \end{bmatrix}$$

So we have the system

$$1 = c_1 + c_2$$
$$0 = c_2 + c_3$$
$$0 = -c_1$$
$$1 = c_2$$

But this has no solution, Therefore our equation above has no solution and therfore

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin \text{span}(S)$$

Definition 63: A subset S of a vector space V is called **linearly dependent** (LD) if there exist finitely many distinct vectors $u_1, ..., u_n \in S$ and scalars $a_1, ..., a_n$ not all zero so that

$$\sum_{i=1}^{n} a_i u_i = 0$$

Alternatively, if the only representation of 0 as a linear combination of the vectors in S is the trivial one, then S is **linearly independent** (LI).
Factoids:

1. $\emptyset$ is LI.

2. A set consisting of exactly one vector in LI.

Theorem 64: Let V be a vector space and let $S_1 \subseteq S_2 \subseteq V$. Then if S_1 is LD, then S_2 is LD, and if S_2 is LI, then S_1 is LI.
Theorem 65: Let S be a LI subset of a vector space V, and let $v \in V \ni v \notin S$, then $S \cup \{v\}$ is LD if and only if $v \in \text{span}(S)$.
Definition 66: A **basis** β for a vector space V is a LI generating set for V.
Example 67: Some examples:

1. $\beta := \{\vec{e}_1, ..., \vec{e}_n\}$ is a basis for $\mathbb{R}^n$, where $\vec{e}_i$ has i^{th} component 1 and all others 0.

2. $\beta := \{1, x, x^2, ..., x^n\}$ is a basis for $\mathbb{P}_n(\mathbb{R})$.

3. $\beta := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M_{2\times 2}(\mathbb{R})$.

Theorem 68: Let V be a vector space and $\beta = \{u_1, ..., u_n\} \subseteq V$. Then β is a basis for V if and only if each $v \in V$ can be uniquely expressed in the form

$$v = a_1 u_1 + ... + a_n u_n$$

for some $a_i \in \mathbb{R}, 1 \leq i \leq n$.

Theorem 69: If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence, V has a finite basis.

Theorem 70 (Replacement Theorem): Let V be a vector space generated by a set G containing exactly n vectors, and let L be an LI subset of V containing exactly m vectors. Then $m \leq n$ and $\exists H \subseteq G \ni H$ contains exactly $n - m$ vectors and $L \cup H$ generates V.

Corollary 71: If you let V be a vector space with finite basis, then every basis for V has the same number of vectors.

Definition 72: A vector space is called **finite dimensional** if it has a basis consisting of a finite number of vectors. A vector space that does not satisfy this is **infinite dimensional**. The unique number of vectors in each basis for V is called the **dimension** of V, denoted $\dim(V)$.

Example 73: Another laundry list:

1. $\dim(\{0\}) = 0$
2. $\dim(\mathbb{R}^n) = n$
3. $\dim(\mathbb{P}_n(\mathbb{R})) = n + 1$
4. $\dim(M_{m \times n}(\mathbb{R})) = mn$

Example 74: Observe that the set

$$S = \{(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1), (0, 0, 0, 1)\} \subseteq \mathbb{R}^4$$

is LI. So we get by Corollary 75 that S generates $\mathbb{R}^4$ and thus is a basis for $\mathbb{R}^4$.

Corollary 75: Let V be a vector space with dimension n.

1. Any finite generating set for V has at least n vectors, and a generating set with exactly n vectors is a basis.
2. Any LI subset of V with exactly n vectors is a basis for V.
3. Every LI subset of V can be extended to a basis for V.

Theorem 76: Let W be a subspace of a finite dimensional vector space V, then W is finite dinensional and $\dim(W) \leq \dim(V)$. If $\dim(W) = \dim(V)$, then $W = V$.

Example 77: Find a basis for the set of solutions to

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_1 - 3x_2 + x_3 = 0$$

Use Gaussian elimination to solve the system.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -3 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 & -2 & 0 \\ 2 & -3 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 & -2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

So we get

$$x_1 = x_3$$
$$x_2 = x_3$$

Now let $t = x_3$, where $t \in \mathbb{R}$. Then

$$x_1 = t$$
$$x_2 = t$$
$$x_3 = t$$

Or in other words $t(1,1,1)$. Therefore a basis for the set of solutions is $\{(1,1,1)\}$.

Platform $9\frac{3}{4}$ (Wronskian and the Column Space):

Definition 78: A set of vector functions $\{v_1(t), ..., v_n(t)\}$ in a vector space V is **linearly independant** on an interval I if for each $t \in I$ the only solution to

$$\sum_{i=1}^{n} c_i v_i(t) = 0$$

is the trivial one ($c_1 = ... = c_n = 0$). If $\exists t_0 \in I \ni \sum_{i=1}^{n} c_i v_i(t_0) = 0$ has a nontrivial solution, then the set is **linearly dependant**.

Definition 79: Given a set of functions $\{f_i(t)\}_{i=1}^{n}$ where $f_i : I \subseteq \mathbb{R} \to \mathbb{R}$, for each $i = 1, ..., n$, then

$$W[f_1, ..., f_n](t) := \begin{vmatrix} f_1(t) & \cdots & f_n(t) \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)}(t) & \cdots & f_n^{(n-1)}(t) \end{vmatrix}$$

is called the **Wronskian** of $f_1, ..., f_n$.

Theorem 80: If $W[f_1, ..., f_n](t) \neq 0, \forall t \in I$, where $f_i : I \subseteq \mathbb{R} \to \mathbb{R}, i = 1, ..., n$, then $\{f_1, ..., f_n\}$ are linearly independant of I.

Remark: If $W[f_1, ..., f_n](t) \equiv 0$ on I, then $\{f_1, ..., f_n\}$ need not be linearly independant.

Example 81:

$$S = \{t^2 + 1, t^2 - 1, 2t + 5\}$$

$$W[f_1, f_2, f_3](t) \begin{vmatrix} t^2+1 & t^2-1 & 2t+5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix} = -8 \neq 0$$

Aside (Column Vectors): Say I have the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 7 \end{bmatrix}$$

Then I can treat each column of the matrix as a vector. Specifically, the vectors (1,0,2), (0,1,4), and (0,0,7).

Definition 82: For any $m \times n$ matrix A, the **column space**, denoted Col(A), is the span of the column vectors of A.

Factoids:

1. The pivot columns of A are a basis for Col(A).
2. dim(Col(A)) =:rank(A).
3. A^{-1} exists if and only if rank(A) $= n$.
4. The column space is a subspace.

Platform $9\frac{3}{4}$ (Inro to Chapter 4): Say I have the DE

$$a_n(t)y^{(n)}(t) + ... + a_1(t)y'(t) + a_0(t)y(t) = f(t)$$

This is hard if the a_n's are actually dependant on time. So let's look at an easier situation, say

$$ay'' + by' + cy = 0$$

By ansatz, we guess that y will have the form $y = e^{\alpha t}$. Therefore

$$y = e^{\alpha t}$$
$$y' = \alpha e^{\alpha t}$$
$$y'' = \alpha^2 e^{\alpha t}$$

Plugging these into the DE we get

$$a\alpha^2 e^{\alpha t} + b\alpha e^{\alpha t} + ce^{\alpha t} = 0$$
$$e^{\alpha t}[a\alpha^2 + b\alpha + c] = 0$$

Since $e^{\alpha t}$ is not zero unless $\alpha = 0$

$$a\alpha^2 + b\alpha + c = 0$$
$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

So we have two possible values for α:

$$\alpha_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
$$\alpha_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

And we always have two solutions:

$$y_1 = e^{\alpha_1 t}$$
$$y_2 = e^{\alpha_2 t}$$

And this works in general.

Aside: The equation

$$a\alpha^2 + b\alpha + c = 0$$

is known as the **characteristic equation** for this type of system.

Example 83:

$$y'' + 6y' + 5y = 0, y(0) = 0, y'(0) = 1$$

By our work above, we get to

$$\alpha^2 + 6\alpha + 5 = 0$$
$$(\alpha + 5)(\alpha + 1) = 0$$

$$\alpha = \{-5, 1\}$$

And so a basis for the solution space is $S = \{e^{-5t}, e^{-t}\}$. Therefore, the solution is

$$y(t) = c_1e^{-5t} + c_2e^{-t}$$

And now solving for the constants we get

$$y(0) = c_1 + c_2 = 0$$
$$v'(0) = -5c_1 - c_2 = 1$$

Example 84:

$$y'' + 2y' + y = 0$$

$$\alpha^2 + 2\alpha + 1 = 0$$
$$(\alpha + 1)^2 = 0$$

$$\alpha_1 = \alpha_2 = -1$$

For some unknown reason, $\gamma = \{e^{-t}, te^{-t}\}$ is a basis for the solution space and therefore the solution is

$$y(t) = c_1e^{-t} + c_2te^{-t}$$

because you can just multiply by a t to get a second solution (and you need a second solution to form your basis).

Example 85:

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0$$

The characteristic equation here is

$$r^4 + 4r^3 + 6r^2 + 4r + 1 = 0$$
$$(r + 1)^4 = 0$$

$$r_1 = r_2 = r_3 = r_4 = -1$$

But we need 4 parts in our basis so we make it $\gamma = \{e^{-t}, te^{-t}, t^2e^{-t}, t^3e^{-t}\}$ and therefore our solution is

$$y(t) = c_1e^{-t} + c_2te^{-t} + c_3t^2e^{-t} + c_4t^3e^{-t}$$

Definition 86: The equation

$$at^2y'' + bty' + cy = 0$$

is called a (homogeneous) **Euler-Cauchy problem**.

Example 87:

$$t^2y'' + 2ty' - 12y = 0$$

Ansatz: $y = t^r$

$$\begin{aligned} y &= t^r \\ y' &= rt^{r-1} \\ y'' &= r(r-1)t^{r-2} \end{aligned}$$

Plugging these in, we get

$$\begin{aligned} t^2(r(r-1)t^{r-2}) + 2t(rt^{r-1}) - 12t^r &= 0 \\ t^r[r(r-1) + 2r - 12] &= 0 \end{aligned}$$

So our characteristic equation is

$$\begin{aligned} r^2 + r - 12 &= 0 \\ (r-3)(r+4) &= 0 \end{aligned}$$

$$r = \{3, -4\}$$

So our basis is $\gamma = \{t^3, t^{-4}\}$ and therefore our solution is

$$y(t) = c_1t^3 + c_2t^{-4}$$

Aside: If $r = r_1 = r_2$, then

$$y(t) = c_1t^r + c_2t^r \ln t$$

Platform $9\frac{3}{4}$ (Order Reduction):

$$y'' + p(t)y' + q(t)y = 0$$

Say I know one solution to this DE, call it $y_1(t)$ and I want to find a second, LI solution. Let $y_2 := v(t)y_1(t)$. Then

$$\begin{aligned} y_2 &= v(t)y_1(t) \\ y_2' &= v'y_1 + vy_1' \\ y_2'' &= v''y_1 + v'y_1' + v'y_1' + vy_1'' \\ &= v''y_1 + 2v'y_1' + vy_1'' \end{aligned}$$

Plug that into the DE and you get

$$\begin{aligned} 0 &= v''y_1 + 2v'y_1' + vy_1'' + p(t)v'y_1 + p(t)vy_1' + q(t)vy_1 \\ &= v''y_1 + 2v'y_1' + pv'y_1 + v[y_1'' + py_1' + qy_1] \\ &= v''y_1 + 2v'y_1' + pv'y_1 \end{aligned}$$

Therefore

$$v''y_1 + [2y_1' + py_1]v' = 0$$

Let $w(t) := v'$. Then

$$\begin{aligned} w'y_1 + [2y_1' + py_1]w &= 0 \\ w' + \frac{2y_1' + py_1}{y_1}w &= 0 \end{aligned}$$

Solve this by integrating factor method.

$$\begin{aligned} \mu(t) &= e^{\int \frac{2y_1' + py_1}{y_1} dt} \\ &= e^{\int \frac{2y_1}{y_1} dt + \int p dt} \\ &= e^{2 \ln |y_1|} e^{\int p dt} \end{aligned}$$

And there are a bunch more steps and you get

$$y_2(t) = y_1(t) \int \frac{e^{-\int p(t) dt}}{{y_1}^2(t)} dt$$

Looking back up at the above steps, note that if $v(t) \equiv C$, this does not work. Thus the two solutions are linearly independant.

Theorem 88 (Reduction of Order): Let y_1 be a known solution on interval $I \subseteq \mathbb{R}$ to the ODE

$$y'' + p(t)'y + q(t)y = 0$$

where $p, q \in C(I)$. Then $B := \{y_1, y_2\}$ is a basis for the solution space, where

$$y_2(t) := y_1(t) \int \frac{e^{-\int p(t)dt}}{{y_1}^2(t)} dt$$

Example 89:

$$y'' + 2y' + y = 0$$

$$\alpha^2 + 2\alpha + 1 = 0$$
$$(\alpha + 1)^2 = 0$$

$$\alpha_1 = \alpha_2 = -1$$

So one solution is

$$y_1 = e^{-t}$$

Using Reduction of Order, we get

$$\begin{aligned} y_2 &= e^{-t} \int \frac{e^{-\int 2dt}}{e^{-2t}} dt \\ &= e^{-t} \int \frac{e^{-2t}}{e^{-2t}} dt \\ &= e^{-t} \int 1dt \\ &= e^{-t}t \end{aligned}$$

Example 90 (Chebyshev's Equation): Find a second solution to

$$(1 - t^2)y'' - ty' + y = 0$$

Given that $y_1(t) = t$ is a solution.

$$\begin{aligned} y_2 &= t \int \frac{e^{-\int -\frac{t}{1-t^2} dt}}{t^2} dt \\ &= t \int \frac{e^{-\frac{1}{2} \ln |1-t^2|}}{t^2} dt \\ &= t \int \frac{1}{t^2\sqrt{1-t^2}} dt \end{aligned}$$

And so our general solution is

$$y(t) = c_1 t + c_2 t \int_{t_0}^{t} \frac{1}{\tau^2\sqrt{1-\tau^2}} d\tau$$

Platform $9\frac{3}{4}$: Say we solve something and our basis is

$$B = \{e^{(\alpha+\beta i)t}, e^{(\alpha-\beta i)t}\}$$

We usually rewrite it like this:

$$\begin{aligned}
y(t) &= k_1 e^{(\alpha+\beta i)t} + k_2 e^{(\alpha-\beta i)t} \\
&= k_1[e^{\alpha t}(\cos\beta t + i\sin\beta t)] + k_2[e^{\alpha t}(\cos\beta t - i\sin\beta t)] \\
&= k_1[e^{\alpha t}(\cos\beta t + i\sin\beta t)] + k_2[e^{\alpha t}(\cos(-\beta t) + i\sin(-\beta t))] \\
&= e^{\alpha t}[(k_1+k_2)\cos\beta t + (k_1-k_2)i\sin\beta t] \\
&= e^{\alpha}t[c_1\cos\beta t + c_2\sin\beta t]
\end{aligned}$$

where

$$\begin{aligned}
c_1 &:= k_1 - k_2 \\
c_2 &:= (k_1 - k_2)i
\end{aligned}$$

Example 91:

$$y'' + 9y = 0$$

Our characteristic equation is

$$r^2 + 9 = 0$$

$$r = \pm 3i = 0 \pm 3i$$

So

$$\begin{aligned}
r_1 &= 0 + 3i \\
r_2 &= 0 - 3i
\end{aligned}$$

$$\begin{aligned}
y(t) &= e^{0t}[c_1\sin 3t + c_2\cos 3t] \\
&= c_1\sin 3t + c_2\cos 3t
\end{aligned}$$

Platform $9\frac{3}{4}$ (Inhomogeneous Equations):

$$ay'' + by' + cy = f(t)$$

Example 92:

$$y'' + 4y' = t$$

Taking a guess at the form of the solution, we have

$$
\begin{aligned}
y_p &= At^2 + Bt + C \\
y_p' &= 2A + B \\
y_p'' &= 2A
\end{aligned}
$$

Plugging these back in we have

$$
\begin{aligned}
2A + 4(2At + B) &= t \\
2A + 8At + 4B &= t \\
8At + (2A + 4B) &= t
\end{aligned}
$$

$$
\begin{aligned}
8A &= 1 \\
2A + 4B &= 0
\end{aligned}
$$

$$
\begin{aligned}
A &= \frac{1}{8} \\
B &= -\frac{1}{16}
\end{aligned}
$$

Notice that the C dropped out because there was no y term in the DE, so it can be anything and so we might as well make it 0. So

$$y_p(t) = \frac{1}{8}t^2 - \frac{1}{16}t$$

and since the general solution to the homogeneous equation

$$y'' + 4y' = 0$$

is found from the characteristic equation

$$r^2 + 4r = 0$$

$$
\begin{aligned}
r_1 &= 0 \\
r_2 &= -4
\end{aligned}
$$

to be

$$y_h(t) = c_1 e^{0t} + c_1 e^{-4t}$$

our final solution is

$$y(t) = y_h + y_p = c_1 + c_2 e^{0t} + \frac{1}{8}t^2 - \frac{1}{16}t$$

Example 93:

$$y'' - y' - 2y = 6e^t$$

The homogeneous equation

$$y'' - y' - 2y = 0$$

has characteristic equation

$$r^2 - r - 2 = 0$$
$$(r-2)(r+1) = 0$$

and has general solution

$$y_h(t) = c_1 e^{2t} + c_2 e^{-t}$$

Our ansatz for the particular solution is

$$y_p = y_p' = y_p'' = Ae^t$$

$$Ae^t - Ae^t - 2Ae^t = 6e^t$$
$$Ae^t = -3Ae^t$$
$$A = -3$$

So

$$y_p = -3e^t$$

and

$$y(t) = y_h + y_p = -3e^t + c_1 e^{2t} + c_2 e^{-t}$$

Example 94:

$$y'' - 2y' + y = 3e^t$$

First solve the homogeneous equation

$$y'' - 2y' + y = 0$$

which has characteristic equation

$$r^2 - 2r + 1 = 0$$
$$(r-1)^2 = 0$$

Thus

$$r_1 = r_2 = 1$$

the general solution is

$$y_h(t) = c_1 e^t + c_2 t e^t$$

and our basis is

$$B = \{e^t, te^t\}$$

For the particular solution, guess

$$\begin{aligned} y_p &= At^2 e^t \\ y_p' &= 2Ate^t + At^2 e^t \\ y_p'' &= 2Ae^t + 4Ate^t + At^2 e^t \end{aligned}$$

Plugging all these in, you find that

$$\begin{aligned} 2Ae^t &= 3e^t \\ A &= \frac{3}{2} \end{aligned}$$

And so

$$y_p(t) = \frac{3}{2} t^2 e^t$$

and therefore the final solution is

$$y(t) = y_h + y_p = c_1 e^t + c_2 t e^t + \frac{3}{2} t^2 e^t$$

Platform $9\frac{3}{4}$ (Variation of Parameters/Constants):

$$y'' + p(t)y' + q(t)y = f(t)$$

The general solution to the forced equation is always going to be

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t)$$

where $B = \{y_1, y_2\}$ is a basis for the solution space. We claim that the particular solution to the original equation will be in the form

$$\begin{aligned} y_p &= v_1 y_1 + v_2 y_2 \\ y_p' &= v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2' = v_1 y_1' + v_2 y_2' \\ y_p'' &= v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2'' \end{aligned}$$

To make this work we require $v_1'y_1 + v_2'y_2 = 0$. Now we plug these in to the original equation to get

$$\begin{aligned} v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2'' + p[v_1y_1' + v_2y_2'] + q[v_1y_1 + v_2y_2] &= f(t) \\ v_1[y_1'' + py_1' + qy_1] + v_2[y_2'' + py_2' + qy_2] + v_1'y_1' + v_2'y_2' &= f(t) \\ v_1'y_1' + v_2'y_2' &= f(t) \end{aligned}$$

Now between this equation and our requirement from above, we get a system of two equations in two variables.

$$\begin{aligned} v_1'y_1' + v_2'y_2' &= f(t) \\ v_1'y_1 + v_2'y_2 &= 0 \end{aligned}$$

which can be rewritten with matrices as

$$\begin{bmatrix} y_1' & y_2' \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

Remark:

$$\begin{aligned} v_1' &= -\frac{y_2 f}{W[y_1, y_2]} \\ v_2' &= -\frac{y_1 f}{W[y_1, y_2]} \end{aligned}$$

Example 95:

$$y'' + y = 4\sin t$$

The homogeneous equation is

$$y'' + y = 0$$

and has cahracteristic equation

$$r^2 + 1 = 0$$

so

$$r = \pm i$$

and its general solution is

$$y_h(t) = c_1 \sin t + c_2 \cos t$$

To find a particular solution to the original equation we first find the Wronskian

$$W[\sin t, \cos t] = \begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = -1$$

Thus

$$v_1' = \frac{-\cos t \cdot 4\sin t}{-1} = 4\sin t\cos t$$
$$v_2' = \frac{4\sin^2 t}{-1} = -4\sin^2 t = -4\frac{1-\cos 2t}{2} = 2\cos 2t - 2$$

and integrating we have

$$v_1(t) = -2\cos^2 t$$
$$v_2(t) = \sin 2t - 2t$$

Platform $9\frac{3}{4}$ (Springs):

$$m\ddot{x} + b\dot{x} + kx = F_0\cos\omega_f t, x(0) = x_0, \dot{x}(0) = v_0$$

Assume $b = 0$. Then we get

$$m\ddot{x} + kx = F_0\cos\omega_f t$$

The forced equation

$$m\ddot{x} + kx = 0$$
$$\ddot{x} + \frac{k}{m}x = 0$$

has characteristic equation

$$r^2 + \frac{k}{m} = 0$$

so

$$r = \pm i\sqrt{\frac{k}{m}}$$

and our general solution is

$$x(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t$$

where $\omega_0 := \sqrt{\frac{k}{m}}$. Now we have two cases.

Case I: $\omega_0 \neq \omega_f$ Since we only have a zeroth and a second derivative, we pick

$$x_p = A\cos\omega_f t + B\sin\omega_f t$$
$$\ddot{x}_p = -A{\omega_f}^2\cos\omega_f t - B{\omega_f}^2\sin\omega_f t$$

Plugging these in we get

$$m(-A{\omega_f}^2\cos\omega_f t - B{\omega_f}^2\sin\omega_f t) + k(A\cos\omega_f t + B\sin\omega_f t) = F_0\cos\omega_f t$$
$$A(k - m{\omega_f}^2)\cos\omega_f t + B(k - {\omega_f}^2)\sin\omega_f t = F_0\cos\omega_f t$$

$$B(k - {\omega_f}^2) = 0$$

so that drops out. Meanwhile

$$A(k - m{\omega_f}^2) = F_0$$

$$A = \frac{F_0}{m(\frac{k}{m} - {\omega_f}^2)} = \frac{F_0}{{\omega_0}^2 - {\omega_f}^2}$$

And so

$$x_p(t) = \frac{F_0}{m({\omega_0}^2 - {\omega_f}^2)} \cos \omega_f t$$

Case II: $\omega_0 = \omega_f$ After a lot of work you end up getting

$$x_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

Platform $9\frac{3}{4}$ (Series Methods):

$$\begin{aligned} a_2(x)y'' + a_1(x)y' + a_0(x)y &= 0 \\ y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y &= 0 \\ y'' + P(x)y' + Q(x)y &= 0 \end{aligned}$$

where

$$\begin{aligned} P(x) &:= \left(\frac{a_1}{a_2}\right)(x) \\ Q(x) &:= \left(\frac{a_0}{a_2}\right)(x) \end{aligned}$$

Definition 96: A function is **analytic** at $x = a$ if it can be represented by a power series centered at $x = a$.
Definition 97: A point is said to be an **ordinary point** of

$$y'' + P(x)y' + Q(x)y = 0$$

if both $P(x)$ and $Q(x)$ are analytic at the point. A point that is not ordinary for

$$y'' + P(x)y' + Q(x)y = 0$$

is called a **singular point**.
Example 98:

$$y'' + y = 0$$

Let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

Plugging these in we get

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$
$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$
$$\sum_{n=0}^{\infty} x^n [a_{n+2}(n+2)(n+1) + a_n] = 0$$

And so our recurrence relation is

$$a_{n+2}(n+2)(n+1) + a_n = 0$$

Solving for a_{n+2}, we get

$$a_{n+2} = -\frac{a_n}{(n+1)(n+2)}, n \geq 0$$

Assume a_0, a_1 given. Then

$$a_2 = -\frac{a_0}{(1)(2)} = -\frac{a_0}{2!}$$
$$a_3 = -\frac{a_1}{(2)(3)} = -\frac{a_1}{3!}$$
$$a_4 = -\frac{a_2}{(3)(4)} = -\frac{[-\frac{a_0}{2!}]}{(3)(4)} = \frac{a_0}{4!}$$
$$a_5 = ... = \frac{a_1}{5!}$$
$$a_6 = ... = -\frac{a_0}{6!}$$
$$1_7 = ... = -\frac{a_1}{7!}$$

and so on. Therefore,

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{a_1}{5!}x^5 + ... \\ &= a_0 \left[a - \frac{x^2}{2!} + \frac{x^4}{4!} - ... \right] + a_1 \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... \right] \\ &= a_0 \cos x + a_1 \sin x \end{aligned}$$

Example 99 (Ary's Equation):

$$y'' - xy = 0$$

Plugging the generalized power series representations of y'' and y as derived above into this equation, we get

$$\begin{aligned} \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} &= 0 \\ \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} a_{n-1} x^{n-1} &= 0 \\ a_2(2)(2-1)x^0 + \sum_{n=2}^{\infty} a_{n+1}(n+1)(n)x^{n-1} - \sum_{n=2}^{\infty} a_{n-1} x^{n-1} &= 0 \\ 2a_2 + \sum_{n=2}^{\infty} x^{n-1}[a_{n+1}n(n+1) - a_{n-2}] &= 0 \end{aligned}$$

And so we get that

$$\begin{aligned} a_2 &= 0 \\ a_{n+1}(n)(n+1) - a_{n-2} &= 0 \end{aligned}$$

Solving the recurrence relation for a_{n+1}, we get

$$\begin{aligned} a_2 &= 0 \\ a_{n+1} &= \frac{a_{n-2}}{(n)(n+1)} \end{aligned}$$

Now, with a_0, a_1 given,

$$
\begin{aligned}
a_2 &= 0 \\
a_3 &= \frac{a_0}{(2)(3)} \\
a_4 &= \frac{a_1}{(3)(4)} \\
a_5 &= 0 \\
a_6 &= \frac{a_3}{(5)(6)} = \frac{a_0}{(2)(3)(5)(6)} \\
a_7 &= ... = \frac{a_1}{(3)(4)(6)(7)} \\
a_8 &= 0
\end{aligned}
$$

and so on. Therefore,

$$
\begin{aligned}
y(x) &= a_0 + a_1 x + \frac{a_0}{(2)(3)} x^3 + \frac{a_1}{(3)(4)} x^4 + ... \\
&= a_0 \left[1 + \frac{x^3}{6} + ... \right] + a_1 \left[a + \frac{x^4}{12} + ... \right]
\end{aligned}
$$

Theorem 100 (Power Series Method): If $x = x_0$ is an ordinary point of

$$y'' + P(x)y' + Q(x)y = 0$$

then there always exist two LI power series solution centered at x_0. The series converge at least on the interval $|x - x_0| < R$, where R is the distance from x_0 to the nearest singularity point.

Example 101:

$$y'' + y = 0, y(0) = 0, y'(0) = 1$$

Put

$$
\begin{aligned}
x_1 &= y \\
x_2 &= y'
\end{aligned}
$$

which is the **canonical substitution**. So

$$
\begin{aligned}
x_1' &= x_2 \\
x_2' &= y''
\end{aligned}
$$

But then

$$
\begin{aligned}
x_1' &= x_2, x_1(0) = 0 \\
x_2' &= -x_1, x_2(0) = 1
\end{aligned}
$$

Platform $9\frac{3}{4}$:

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_2y'' + a_1y' + a_0y = f(t)$$

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ x_3 &= y'' \\ \vdots &= \vdots \\ x_j &= y^{(j-1)} \\ \vdots &= \vdots \\ x_{n-1} &= y^{(n-2)} \\ x_n &= y^{(n-1)} \end{aligned}$$

and

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ \vdots &= \vdots \\ x_{n-1}' &= x_n \end{aligned}$$

and

$$x_n' = y^{(n)} = f(t) - a_{n-1}y^{(n-1)} - ... - a_0y$$

Note that

$$x_n' = f(t) - \sum_{j=0}^{n-1} a_j x_{j+1}$$

and you can write all of this in matrix vector form as

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_{n-1}' \\ x_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{bmatrix}$$

Example 102:

$$\begin{aligned} x_1' &= -2x_1 + x_2 \\ x_2' &= x_1 - 2x_2 \end{aligned}$$

or alternately

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

From the first equation we have

$$x_2 = x_1' + 2x_1$$
$$x_2' = x_1'' + 2x_1'$$

and from the second equation we have

$$(x_1'' + 2x_1') = x_1 - 2(x_1' + 2x_1)$$
$$x_1'' + 2x_1' = x_1 - 2x_1' - 4x_1$$

$$x_1'' + 4x_1' + 3x_1 = 0$$

So our characteristic equation is

$$r^2 + 4r + 3 = 0$$

and therefore

$$r_1 = 1$$
$$r_2 = 3$$

and so

$$\begin{aligned} x_2 &= \frac{d}{dt}\left(c_1 e^{-3t} + c_2 e^{-t}\right) + 2\left(c_1 e^{-3t} + c_2 e^{-t}\right) \\ &= -c_1 e^{-3t} + c_2 e^{-t} \end{aligned}$$

$$x_1(t) = c_1 e^{-t} + c_2 e^{-3}$$
$$x_1(t) = c_1 e^{-t} - c_2 e^{-3}$$

or, in matrix vector form,

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}$$

Platform $9\frac{3}{4}$ (Aigen Vectors): From above, take the matrix

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

and let it be part of the definition of a linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$,

$$T\vec{v} := \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \vec{v}$$

Now, note that

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$T\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = (-3)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So we see that

$$T\vec{v} = \lambda\vec{v}$$

for λ taken from the exponents and $\vec{v}$ is taken from that final matrix vector form of our solution. To prove this note that

$$\begin{aligned} \vec{x} &= c_1\vec{v}_1e^{\lambda_1 t} + c_2\vec{v}_2e^{\lambda_2 t} \\ \vec{x}' &= c_1\vec{v}_1\lambda_1e^{\lambda_1 t} + c_2\vec{v}_2\lambda_2e^{\lambda_2 t} \\ &= c_1A\vec{v}_1e^{\lambda_1 t} + c_2A\vec{v}_2e^{\lambda_2 t} \\ &= A[c_1\vec{v}_1e^{\lambda_1 t} + c_2\vec{v}_2e^{\lambda_2 t}] \\ &= A\vec{x} \end{aligned}$$

and the random substitution in the middle there is

$$\begin{aligned} A\vec{v}_1 &= \lambda_1\vec{v}_1 \\ A\vec{v}_2 &= \lambda_2\vec{v}_2 \end{aligned}$$

Definition 103: Let V and W be vector spaces over a field F. We call a function $T : V \to W$ a **linear transformation** from V into W if, $\forall x, y \in V$ and $c \in F$, we have:

1. $T(x + y) = Tx + Ty$
2. $T(cx) = cTx$

Proposition 104: Given any linear transformation $T : V \to W$, the following hold:

1. $T\vec{0}_V = \vec{0}_W$
2. $T(cx + y) = cTx + ty, \forall x, y \in V, c \in F$

3. $T(x-y) = Tx - Ty, \forall x, y \in V$

4. $T(\sum_{i=1}^{n} c_i x_i) = \sum_{i=n}^{n} c_i T x_i, \{x_i\}_{i=1}^{n} \subseteq V, \{c_i\}_{i=1}^{n} \subseteq F$

Proof:

1.

$$\begin{aligned} T(0+x) &= Tx \\ T(0+x) &= T(0) + Tx \\ Tx &= T(0) + Tx \\ 0 &= T(0) \end{aligned}$$

2. If T is linear, then

$$T(cx+y) = T(cx) + Ty = cTx + Ty$$

3. Just let $c = -1$ and use Property 2 from Definition 103.

4. Use induction.

Example 105:

$$y' = y, y(t) = 0$$

By the FTC,

$$y(t) = \int_{t_0}^{t} y(s)ds$$

Define a linear operator

$$Ty := \int_{t_0}^{t} y(s)ds$$

It is really easy to prove that this is linear and I'm too lazy to type it out for you.

Definition 112: Let V and W be vector spaces. Let $T : V \to W$ be linear. If $N(T) = \ker(T)$ and $R(T)$ are finite-dimensional, then **nullity**$(T) := \dim(N(T))$, and **rank**$(T) := \dim(R(T))$.

Example 113:

$$T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R})$$

$$T(f(x)) = xf(x) + f'(x)$$

Note that $\gamma := \{1, x, x^2\}$ is a basis for $\mathbb{P}_2(\mathbb{R})$.

$$\begin{aligned} T(1) &= x \cdot 1 + (1)' = x \\ T(x) &= x \cdot x + (x)' = x^2 + 1 \\ T(x^2) &= x \cdot x^2 + (x^2)' = x^3 + 2x \end{aligned}$$

By Theorem 111, $\beta := \{x, x^2+1, x^3+2x\}$ is a basis for $R(T)$.

$$T(f(x)) = 0 \Leftrightarrow xf + f'(x) = 0 \Leftrightarrow f \equiv 0$$

Therefore, $\ker(T) = \{0\}$ and therefore $\text{nullity}(T) = 0$. Also, $\text{rank}(T) = 3$.

Theorem 114: Let $T : V \to W$ be linear. Then T is one-to-one if and only if $N(T) = \{0\}$ (that is, $\text{nullity}(T) = 0$).

Proof: In the forward direction: Suppose T is one-to-one and $x \in N(T)$. Then $Tx = 0 = T(0)$. Thus, $N(T) \subseteq \{0\}$. Since T is injectove, $x = 0$; so $N(T) \supseteq \{0\}$ and so $N(T) = \{0\}$. In the reverse direction: Suppose that $N(T) = \{0\}$. Say $Tx = Ty$ for some $x, y \in V$. Claim that $x = y$. Note then that $Tx = Ty \Rightarrow Tx - Ty = 0 \Rightarrow T(x-y) = 0 \Rightarrow x - y = 0 \Rightarrow x = y$

Theorem 115 (Dimension Theorem): Let $T : V|rightarrowW$ be linear. If V is finite-dimensional, then

$$\text{nullity}(T) + \text{rank}(T) = \dim(V)$$

Theorem 116:

$$T : M_{2\times 3}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$$

$$T\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}\right) = \begin{bmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{bmatrix}$$

Note that $\alpha := \{E_{11}, E_{12}, ..., E_{23}\}$ is a basis for $M_{2\times 3}(\mathbb{R})$ and $\text{card}(\alpha) = 6$.

$$TE_{11} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$TE_{12} = \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$TE_{13} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$TE_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$TE_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$TE_{23} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Theorem 111 implies that

$$R(T) = \text{span}\{TE_{ij}\} = \text{span}\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$$

Therefore

$$\beta := \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

is a basis for $R(T)$.

Platform $9\frac{3}{4}$ (Finding a Basis for $N(T)$):

$$2a_{11} - a_{12} = 0$$
$$a_{13} + 2a_{12} = 0$$

$$a_{11} = -\frac{1}{4}a_{13}$$
$$a_{12} = -\frac{1}{2}a_{13}$$

$$N(T) = \left\{ \begin{bmatrix} -\frac{1}{4}a & -\frac{1}{2}a & a \\ b & c & d \end{bmatrix} \right\}$$

Therefore a basis for $N(T)$ is

$$\left\{ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

Theorem 117: Suppose that $\dim(V) = \dim(W) < \infty$ and $T : V \to W$ is linear. Then the following are equivalent:

1. T is one-to-one
2. T is onto
3. $\text{rank}(T) = \dim(V)$

Example 118:

$$T : \mathbb{P}_2(\mathbb{R} \to \mathbb{P}_3(\mathbb{R}))$$
$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)dt$$

$$T(1) = 2(1)' + \int_0^x 3dt = 3x$$
$$T(x) = 2(x)' + \int_0^x 3tdt = 2 + \frac{3}{2}x^2$$
$$t(x^2) = 2(x^2)' + \int_0^x 3t^2dt = 4x + x^3$$

Therefore as $\{T1, Tx, Tx^2\}$ is a L.I. set, by Theorem 111,

$$\alpha := \{3x, 2 + \frac{3}{2}x^2, 4x + x^3\}$$

is a basis for $R(T)$. Since nullity$(T) = 0$ and $\ker(T) = \{0\}$, by Theorem 115,

$$\begin{aligned} \text{nullity}(T) &= \dim(\mathbb{P}_2(\mathbb{R})) - \text{rank}(T) \\ &= 3 - 3 \\ &= 0 \end{aligned}$$

Also, this means that the unique solution to the integrodifferential equation

$$2y' + \int_0^x 3y(t)dt = 0$$

is $y(t) \equiv 0$.

Theorem 119: Suppose $\{v_1, ..., v_n\}$ is a basis for V. For $w_1, ..., w_n \in W$, there does not exist a linear transformation $T : V \to W \ni Tv_i = w_i$, for $1 \leq i \leq n$.

Corollary 120: Suppose V is finite-dimensional with basis $\{v_1, ..., v_n\}$. If $U, T : V \to W$ are linear and $Uv_i = Tv_i$, for each i, then $U \equiv T$.

Definition 121: Let V be finite-dimensional. An **ordered basis** for V is a basis for V equipped with a specific order.

Example 122:

$$\begin{aligned} \gamma &= \{e_1, e_2, e_3\} \\ \beta &= \{e_1, e_3, e_2\} \end{aligned}$$

Note that $\gamma \neq \beta$ as ordered bases, but $\gamma = \beta$ as unordered bases.

Defnition 123: Let $\beta = \{u_1, ..., u_n\}$ be an ordered basis for V. For $x \in V$, let $a_1, ..., a_n$ be the unique scalars such that $x = \sum_{i=1}^n$. We define the coordinate transform of x relative to β, denoted $[x]_\beta$, by

Example 124: Let $V = \mathbb{P}_4(\mathbb{R})$. $\beta = \{1, x, x^2, x^3, x^4\}$ in an ordered basis for V. Then if $f(x) =$

...

Definition 138: Let $T : V \to W$ be linear. A function $U : W \to V$ is an **inverse** of T if $TU = I_W$ and $UT = T_V$. If T has an inverse, then T is **invertible**. In this case, the inverse is unique and denoted by T^{-1}.

Factoids:

1. $(TU)^{-1} = U^{-1}T^{-1}$
2. $(T^{-1})^{-1} = T$

Theorem 139: Let $T : V \to W$ be linear, where V and W are finite-dimensional vector spaces of equal dimension. Then T^{-1} exists if and only if $\text{rank}(T) = \dim(V)$.

Theorem 140: Let V and W be vector spaces and $T : V \to W$ be linear and invertible. Then $T^{-1} : W \to V$ is linear.

Theorem 141: Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. Let $T : V \to W$ be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. In this case,

$$\left[T^{-1}\right]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$$

Example 142:

$$T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R})$$

$$Tf := f + f' + f''$$

$$\beta = \{1, x, x^2\}$$

$$[T]_\beta = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$
$$([T]_\beta)^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Corollary to Steinauer's Theorem: Let God exist. Goodrich likes to draw pictures and should have been an algebraist. Recall that Jesus would have been an algebraist. Therefore Goodrich $\equiv$ Jesus.

Definition 143: Let V and W be vector spaces. We say that V is **isomorphic** to W, denoted $V \approx W$, if there exists a linear transformation T that maps V into W such that T is invertible. Such a linear transformation is called an **isomorphism** from V onto W.

Theorem 144: Let V and W be finite-dimensional vector spaces over a common field F. Then $V \approx W$ if and only if $\dim(V) = \dim(W)$.

Theorem 145: Let V and W be finite-dimensional vector spaces of dimension n and m, respectively, and let β and γ be oredered bases for V and W, respectively. Then

$$\Phi : \mathcal{L}(V, W) :\to \mathbb{M}_{m \times n}(\mathbb{R})$$

defined by

$$\Phi(T) = [T]_\beta^\gamma$$

is an isomorphism.

Platform $9\frac{3}{4}$ (Eigenstuff):

$$Ty = \lambda y$$

y is the eigenvector and λ is the eigenvalue.

Definition 146: Let $T : V \to V$ be linear. A scalar λ is an **eigenvalue** of T if there is a nonzero $v \in V \ni Tv = \lambda v$. In this case, v is called the **eigenvector** of T associated to λ, and (λ, v) is an **eigenpair**.

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ A\vec{v} - \lambda\vec{v} &= 0 \\ A\vec{v} - \lambda I\vec{v} &= 0 \\ (A - \lambda I)\vec{v} &= 0 \end{aligned}$$

and in order for this to have a nontrivial solution, the coefficient matrix $(A - \lambda I)$ must have a determinant of 0 because if $\det(A - \lambda I) \neq 0$, then the equation above has a unique solution (i.e., $\vec{v} = \vec{0}$). So the characteristic equation of A is

$$\det(A - \lambda I) = 0$$

Example 147:

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det\left(\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) & \\ \begin{vmatrix} 1-\lambda & 3 \\ 1 & 3-\lambda \end{vmatrix} &= 0 \\ 3 - 4\lambda + \lambda^2 - 3 &= 0 \\ \lambda^2 - 4\lambda &= 0 \\ \lambda(\lambda - 4) &= 0 \\ \lambda_1 &= 0 \\ \lambda_2 &= 4 \end{aligned}$$

Now find an eigenvector for $\lambda_1 = 0$:

$$\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + 3v_2 = 0$$
$$v_1 + 3v_2 = 0$$

$$v_1 = -3v_2$$

So a possible eigenvector is

$$\begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

and so our eigenpair is

$$\left(0, \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$$

Now we find an eigenvector for $\lambda_2 = 4$:

$$(A - 4I)\vec{v} = 0$$
$$\begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3v_1 + 3v_2 = 0$$
$$v_1 - v_2 = 0$$

$$v_1 = v_2$$

So a possible eigenvector is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and so our eigenpair is

$$\left(4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

Theorem 148 (Eigenspace Theorem): For each eigenvalue λ of a linear transformation $T : V \to V$, the eigenspace

$$\mathbb{E}_\lambda := \{\vec{v} \in V : T\vec{v} = \lambda\vec{v}\}$$

is a subspace of V.

Theorem 149 (Distinct Eigenvalue Theorem): Let $A \in \mathbb{M}_{m\times n}(\mathbb{R})$. If $\lambda_1, ..., \lambda_p$ are distinct eigenvalues with associated eigenvectors $\vec{v}_1, ..., \vec{v}_n$ ($p \leq n$), then $\{\vec{v}_1, ..., \vec{v}_n\}$ is a linearly independant set.

Proposition 150:

1. If A is diagonal, then the eigenvalues of A lie along the main diagonal.

2. If A is upper or lower triangular, then its eigenvalues are on the main diagonal.

3. If $A \in \mathbb{M}_{2\times 2}(\mathbb{R})$, then the characteristic equation of A is

$$\lambda^2 - (\text{Tr}A)\lambda + |A| = 0$$

Example 151: Consider the two-point BVP

$$\begin{aligned} y'' + \lambda y &= 0 \\ y'(0) &= 0 \\ y(\pi) &= 0 \end{aligned}$$

Our characteristic equation is

$$r^2 + \lambda = 0$$

We must consider this case-by-case. First, consider $\lambda = 0$.

$$\begin{aligned} r^2 &= 0 \\ r &= 0 \end{aligned}$$

Therefore

$$y(t) = At + B$$

But

$$y'(0) = 0 = A$$

and

$$y(\pi) = 0 = A\pi + B = B$$

so the only possible $y(t)$ is

$$y(t) \equiv 0$$

but this is the trivial solution so we get no eigenvectors from this case. Now consider $\lambda < 0$.

$$\begin{aligned} r^2 + \lambda &= 0 \\ r^2 &= -\lambda \\ r &= \pm\sqrt{-\lambda} = \pm\mu \end{aligned}$$

Therefore

$$y(t) = Ae^{\mu t} + Be^{-\mu t}$$

but according to Goodrich this doesn't work out either, so we don't get any eigenvectors from this case either. Finally consider $\lambda > 0$.

$$\begin{aligned} r^2 + \lambda &= 0 \\ r^2 &= -\lambda \\ r &= \pm i\sqrt{\lambda} \\ r &= \pm i\mu \end{aligned}$$

Therefore

$$y(t) = c_1 \cos(\mu t) + c_2 \sin(\mu t)$$

and this time it works, because

$$y'(x) = -c_1\mu \sin(\mu x) + c_2\mu \cos(\mu x)$$

So, checking the boundary conditions,

$$y'(0) = 0 = c_2\mu$$

and therefore $c_2 = 0$. Also,

$$y(\pi) = 0 = c_1 \cos(\mu\pi)$$

which is satisfied when

$$\mu = \mu_n := \frac{2n-1}{2}, n \in \mathbb{N}$$

So our eigenvalues are

$$\lambda_n = {\mu_n}^2 = \left(\frac{2n-1}{2}\right)^2, n \in \mathbb{N}$$

and our eigenvectors are

$$y_n(x) = \cos\left(\frac{2n-1}{2}x\right), n \in \mathbb{N}$$

Note that there are infinitely many eigenvalues. Also note that this problem has to do with the heat PDE but I didn't write down the details.

Platform $9\frac{3}{4}$ (Factoring Matrices):

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

First, find the eigenstuff.

$$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

So the eigenvalues are

$$\lambda_1 = 3$$
$$\lambda_2 = -1$$

Now we find some eigenpairs. First, for λ_1:

$$\begin{bmatrix} -2 & 1 & 0 \\ 4 & -2 & 0 \end{bmatrix}$$

So

$$-2v_1 + v_2 = 0$$
$$v_1 = \frac{1}{2}v_2$$

So a possible eigenvector is

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and the assiciated eigenpair is

$$\left(3, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

Now λ_2:

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & 2 & 0 \end{bmatrix}$$

So

$$2v_1 + v_2 = 0$$
$$v_1 = -\frac{1}{2}v_2$$

So a possible eigenvector is

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

and the assiciated eigenpair is

$$\left(-1, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$

Okay now let's take it a step forward. Consider

$$D := \text{diag}\{\lambda_1, \lambda_2\} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$P := \begin{bmatrix} \vdots & \vdots \\ \vec{v}_1 & \vec{v}_2 \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$$

with associated inverse

$$P^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Note that we just factored the matrix since

$$A = PDP^{-1}$$

Now let's switch gears entirely and consider the system of differential equations represented by A:

$$\vec{x}' = A\vec{x}$$

which can be written as

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

or

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= 4x_1 + x_2 \end{aligned}$$

Now do a clever change of variables and put $\vec{x} = P\vec{u}$, which is equivalent to $P^{-1}\vec{x} = \vec{u}$. This means that

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So

$$\begin{aligned} \vec{x}' &= A\vec{x} \\ (P\vec{u})' &= A\vec{x} \\ P\vec{u}' &= A\vec{x} \\ P\vec{u}' &= PDP^{-1}\vec{x} \\ P\vec{u}' &= PD\vec{u} \\ \vec{u}' &= P^{-1}PD\vec{u} \\ \vec{u}' &= ID\vec{u} \\ \vec{u}' &= D\vec{u} \end{aligned}$$

which can be written as

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and

$$u_1' = 3u_1$$
$$u_2' = -u_2$$

And see now this system is really easy to solve. Hooray for clever math tricks involving matrices! Now, we can show this in general.

Theorem 152 (Diagonalization of a Matrix): Assume that $A \in \mathbb{M}_{n\times m}(\mathbb{R})$ is given and has $\{\lambda_i\}_{i=1}^n$ as its n (not necessarily distinct) eigenvalues and $\{\vec{v}_i\}_{i=1}^n$ as its n linearly independent associated eigenvectors. Then if and only if we put

$$P := \begin{bmatrix} \vdots & & \vdots \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \vdots & & \vdots \end{bmatrix}$$

and

$$D := \text{diag}\{\lambda_1, ..., \lambda_n\}$$

then

$$AP = PD$$

i.e.,

$$A = PDP^{-1}$$

Definition 153: A and B are called **similar**, denoted

$$B\ A$$

if

$$B = P^{-1}AP$$

Theorem 154: For a linear DE

$$\vec{x}' = A\vec{x}$$

with diagonizable matrix A, the change of variables $\vec{x} = P\vec{w}$ transforms the original system into a decoupled system

$$\vec{w}' = D\vec{w}$$

where $D = \text{diag}\{\lambda_1, ..., \lambda_n\}$, where $\lambda_1, ..., \lambda_n$ are the eigenvalues of A.

Theorem 155: For a linear DE

$$\vec{x}' = A\vec{x} + f(t)$$

with diagonizable matrix A, the change of variables $\vec{x} = P\vec{w}$ transforms the original system into a decoupled system

$$\vec{w}' = D\vec{w} + P'f$$

where $D = \text{diag}\{\lambda_1, ..., \lambda_n\}$, where $\lambda_1, ..., \lambda_n$ are the eigenvalues of A.

Example 156:

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Platform 9$frac34$ **(The Eigenvalue Method):**

$$\vec{x}' = A\vec{x}$$

A has eigenvalues $\lambda_1, ..., \lambda_n$ and eigenvectors $\vec{v}_1, ..., \vec{v}_n$, we have

$$\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + ... + c_n\vec{v}_n e^{\lambda_n t}$$

and, differentiating,

$$\begin{aligned} vecvecx'(t) &= c_1\vec{v}_1\lambda_1 e^{\lambda_1 t} + ... + c_n\vec{v}_n\lambda_n e^{\lambda_n t} \\ &= c_1 A\vec{v}_1 e^{\lambda_n t} + ... + c_n A\vec{v}_n e^{\lambda_n t} \\ &= A\left[c_1\vec{v}_1 e^{\lambda_n t} + ... + c_n\vec{v}_n e^{\lambda_n t}\right] \\ &= A\vec{x} \end{aligned}$$

Theorem 157 (General Existence and Uniqueness): If $f(t)$ and $A(t)$ are continuous on some interval $I \ni t_0 \in I$, then the IVP

$$\vec{x}' = A(t)\vec{x} + f(t), \vec{x}(t_0) = \vec{x}_0$$

has a unique solution that exists on all of I.

Example 158:

$$\begin{aligned} x' &= -x + 2y, x(0) = 4 \\ y' &= 4x + y, y(0) = -3 \end{aligned}$$

$$\vec{x}' = \begin{bmatrix} -1 & 2 \\ 4 & 1 \end{bmatrix} \vec{x}, \vec{x}(0) = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

Find the eigenstuff:

$$\begin{vmatrix} -1-\lambda & 2 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

So

$$\lambda^2 - 9 = 0$$

and therefore our eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -3$. For λ_1, note that

$$v_1^1 = \frac{1}{2} v_1^2$$

and for λ_2, note that

$$v_2^1 = -v_2^2$$

So we pick

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So we have

$$\begin{aligned} \vec{x}(t) &= c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} \\ &= \begin{bmatrix} c_1 e^{3t} + c_2 e^{-3t} \\ 2c_1 e^{3t} - c_2 e^{-3t} \end{bmatrix} \end{aligned}$$

and, of course, this means that

$$\begin{aligned} x(t) &= c_1 e^{3t} + c_2 e^{-3t} \\ y(t) &= 2c_1 e^{3t} - c_2 e^{-3t} \end{aligned}$$

Now, we go back and solve for initial conditions:

$$\begin{aligned} x(0) = 4 &= c_1 + c_2 \\ y(0) = -3 &= 2c_1 - c_2 \end{aligned}$$

So

$$c_1 = \frac{1}{3}$$

and

$$c_2 = \frac{11}{3}$$

And then we draw a sweet phase portrait.

Example 159:

$$x' = -2x + y$$
$$y' = 2x - y$$

$$\vec{x}' = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \vec{x}$$

Find the eigenstuff:

$$\begin{vmatrix} -2-\lambda & 1 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

So

$$\lambda^2 + 3\lambda = 0$$

and therefore our eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 0$. For λ_1, note that

$$v_1^1 = -v_1^2$$

and for λ_2, note that

$$v_2^1 = \frac{1}{2} v_2^2$$

So we pick

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

So we have

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} c_1 e^{-3t} + c_2 \\ -c_1 e^{-3t} + 2c_2 \end{bmatrix}$$

and, of course, this means that

$$x(t) = c_1 e^{-3t} + c_2$$
$$y(t) = -c_1 e^{-3t} + 2c_2$$

And the phase portrait is really cool.

Platform $9\frac{3}{4}$ (Classifications):

$\lambda_1 > 0 > \lambda_2$: Saddle point: one line repells solutions from the origin, the other attracts.

$0 > \lambda_1 > \lambda_2$: $\vec{v}_1$ is the slow eigendirection, so solutions get closer to it faster than to $\vec{v}_2$, but it still gets to $(0, 0)$.

$\lambda_1 > \lambda_2 > 0$: Same thing as above.

$\lambda_2 < 0 = \lambda_2$: One line ($\vec{v}_2$) is composed entirely of equilibria. The other attracts solutions to the $\vec{v}_2$ line.

Example 160:

$$x' = -2x - 3y$$
$$y' = 3 - 2$$

$$A = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix}$$

$$\begin{vmatrix} -2-\lambda & -3 \\ 3 & -2-\lambda \end{vmatrix} = 0$$

so

$$4 + 4\lambda + 13 = 0$$

And you can go quandratic that yourself, I'm definitely not going to. The answers are

$$\lambda_1 = -2 + 3i$$

and

$$\lambda_2 = -2 - 3i$$

Consider $\lambda_1 = -2 + 3i$ Our nontrivial solution is

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Consider $\lambda_2 = -2 - 3i$ Our nontrivial solution is

$$\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

So our solution is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(-2+3i)} + c_2 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(-2-3i)}$$

Platform $9\frac{3}{4}$ (Davis's Theorem):

$$\text{Gauss} \approx \text{Jesus}$$

Proof: Just think about it.

Example 161:

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

Eigenvalues:

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Eigenvectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1+i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1-i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Solutions:

$$\vec{x}_1(t) = e^{0t} \cos t \begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{0t} \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{0-it} \vec{v}_1$$
$$\vec{x}_2(t) = e^{0t} \cos t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{0t} \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} = e^{0+it} \vec{v}_2$$

General Solution:

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1 \left(\cos t \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + c_2 \left(\cos t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

... (Sally has this derivation) ...

Platform $9\frac{3}{4}$ (Nonlinear Systems):

$$\dot{x} = \sum_{i+j=0}^{n} a_i j x^i y^j$$
$$\dot{y} = \sum_{i+j=0}^{n} b_i j x^i y^j$$

Unsolved Problem 162: What is the maximum number of limit cycle in this equation for each $n \in \mathbb{N}$?
Example 163:

$$x' = x + y - x(x^2 + y^2)$$
$$y' = -x + y - y(x^2 + y^2)$$

...

Theorem 179 (Variation of Constants Formula): Assume that A is a continuous $n \times n$ matrix function on an interval I, $\vec{b}$ is a continuous $n \times 1$ vector function on I, and Φ is a fundamental matrix for $\vec{x} = A(t)\vec{x}$. Then the solution to the IVP

$$\vec{x}' = A(t)\vec{x} + \vec{b}(t), \vec{x}(t_0) = \vec{x}_0$$

where $t_0 \in I$ and $\vec{x}_0 \in \mathbb{R}^n$, is given by

$$\vec{x}(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)\vec{b}(s)ds$$

for $t \in I$.
Proof: First find $\vec{x}'$ and see if it satisfies the IVP:

$$\begin{aligned}
\vec{x}' &= \Phi'(t)\Phi^{-1}(t_0)x_0 + \Phi'(t)\int_{t_0}^{t} \Phi^{-1}(t)\vec{b}(s)ds + \Phi(t)\Phi^{-1}(t)\vec{b}(t) \\
&= \Phi'(t)\Phi^{-1}(t_0)x_0 + A(t)\Phi(t)\int_{t_0}^{t} \Phi^{-1}(t)\vec{b}(s)ds + \vec{b}(t) \\
&= A(t)\Phi(t)\Phi^{-1}(t_0)x_0 + A(t)\Phi(t)\int_{t_0}^{t} \Phi^{-1}(t)\vec{b}(s)ds + \vec{b}(t) \\
&= A(t)[\Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(t)\vec{b}(s)ds] + \vec{b}(t) \\
&= A(t)\vec{x}(t) + \vec{b}(t)
\end{aligned}$$

Next check the initial conditions:

$$\vec{x}(t_0) = \Phi(t_0)\Phi^{-1}(t_0)\vec{x}_0 + \Phi(t)\int_{t_0}^{t_0} \Phi^{-1}(s)\vec{b}(s)ds = \vec{x}_0$$

And we are done.

Corollary 180: Assume that A is an $n \times n$ constant matrix and $\vec{b}$ is a continuous $n \times 1$ vector function on an interval I. Then the solution $\vec{x}(t)$ of the IVP

$$\vec{x}' = A\vec{x} + \vec{b}(t), \vec{x}(t_0) = \vec{x}_0$$

where $t_0 \in I$ and $\vec{x}_0 \in \mathbb{R}^n$, is

$$\vec{x}(t) = e^{A(t-t_0)}\vec{x}_0 + \int_{t_0}^{t} e^{A(t-s)}\vec{b}(s)ds$$

for $t \in I$.

Proof: Put $\Phi := e^{At}$ in Theorem 179. Then using the fact that $(e^{At})^{-1} = e^{-At}$, we find that

$$\begin{aligned}\vec{x}(t) &= \Phi(t)\Phi^{-1}(t_0)\vec{x}_0 + \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)\vec{b}(s)ds \\ &= e^{At}e^{-At_0}\vec{x}_0 + e^{At}\int_{t_0}^{t} e^{-As}\vec{b}(s)ds \\ &= e^{A(t-t_0)}\vec{x}_0 + \int_{t_0}^{t} e^{A(t-s)}\vec{b}(s)ds\end{aligned}$$

Remark: In case $t_0 = 0$:

$$\vec{x}(t) = e^{At}\vec{x}_0 + \int_{0}^{t} e^{A(t-s)}\vec{b}(s)ds$$

Example 181:

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\vec{x} + \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

First find the eigenvalues:

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = 0$$

$$3 - 4\lambda + \lambda^2 + 1 = 0$$

Therefore

$$\lambda_1 = \lambda_2 = 2$$

Now we run through some algorithm:

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} M_1 &= A - \lambda_1 I \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$

and

$$p_1' = 2p_1, p_1(0) = 1$$

so

$$p_1(t) = e^{2t}$$

and

$$\begin{aligned} p_2' = 2p_2 + e^{2t}, p_2(0) = 0 \\ p_2' - 2p_2 = e^{2t} \end{aligned}$$

...

$$e^{-2t}p_2 = \int e^0 dt = t + C$$

therefore

$$p_2 = te^{2t} + Ce^{2t}$$

so

$$p_2(t) = te^{2t}$$

and therefore

$$e^{At} = p_1 M_0 + p_2 M_1 = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix} + \begin{bmatrix} -te^{2t} & te^{2t} \\ -te^{2t} & te^{2t} \end{bmatrix} = \begin{bmatrix} e^{2t} - te^{2t} & te^{2t} \\ -te^{2t} & e^{2t} + te^{2t} \end{bmatrix}$$

...

$$\vec{x}(t) = e^{2t} \begin{bmatrix} 2 + \frac{1}{2}t^2 - t^3 \\ 1 + t - \frac{1}{2}t^2 - t^3 \end{bmatrix}$$

Example 182: Use variation of constants to solve

$$\vec{x}' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 2te^t \\ 0 \\ 0 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Example 183: Consider the system

$$x' = -y + x(x^2+y^2)\sin\left(\frac{\pi}{\sqrt{x^2+y^2}}\right)$$
$$y' = x + y(x^2+y^2)\sin\left(\frac{\pi}{\sqrt{x^2+y^2}}\right)$$

Recall that

$$r\dot{r} = x\dot{x} + y\dot{y}$$

and

$$\dot{\theta}\sec^2\theta = \frac{\dot{y}x + \dot{x}y}{x^2}$$

...

$$\theta' = 1$$

so the limit cycles move clockwise, and

$$r' = r^3 \sin\left(\frac{\pi}{r}\right)$$

so we get limit cycles of radius $r = \frac{1}{k}, k \in \mathbb{N}$. Note that they are of alternating stability.

Platform $9\frac{3}{4}$ (The Laplace Transform): It's basically Transfirguration.

Definition 184: Given a nice function $f(t)$, we define the **Laplace transform** of $f(t)$ by

$$\mathcal{L}\left\{f(t)\right\}(s) := \int_0^\infty e^{-st} f(t)dt =: F(s)$$

Remark: $\mathcal{L}$ is a linear transformation between suitable infinite dimensional vector spaces.

Platform $9\frac{3}{4}$ (Laplace Trasfrom of Heaviside-Type Functions):

$$f(t) = \left\{\begin{array}{ccc} 0 & , & t < a \\ g(t) & , & t > a \end{array}\right\} = H(t-a)g(t)$$

Using the substitution $\tau = t - a$,

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)dt \\
&= \int_a^\infty e^{-st} g(t)dt \\
&= \int_0^\infty e^{-s(\tau+a)} g(\tau + a)d\tau \\
&= e^{-sa}\int_0^\infty e^{-st} g(\tau + a)d\tau \\
&= e^{-sa}\mathcal{L}\{g(t+a)\}
\end{aligned}$$

Proposition 185: Suppose that $a > 0$ and $g(t)$ is nice. Then

$$\mathcal{L}\{g(t)H(t-a)\} = e^{-sa}\mathcal{L}\{g(t+a)\}$$

Platform $9\frac{3}{4}$ (Laplace Transform of $f'(t)$):

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$$

Use integration by parts:

$$\begin{aligned} u &= e^{-st} & dv &= f'(t) \\ du &= -se^{-st} & v &= f(t) \end{aligned}$$

Thus we have

$$\begin{aligned} \mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)dt \\ &= e^{-st} f(t)|_0^\infty + \int_0^\infty se^{-st} f(t)dt \\ &= \left[\lim_{b\to\infty} e^{-sb} f(b)\right] - e^0 f(0) + s\int_0^\infty e^{-st} f(t)dt \\ &= 0 - f(0) + s\mathcal{L}\{f(t)\} \\ &= s\mathcal{L}\{f(t)\} - f(0) \end{aligned}$$

Theorem 186: Assume that f is of exponential order (that is, the limit $\lim_{b\to\infty} e^{-sb} f(b)$ from above exists). Then

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

Platform $9\frac{3}{4}$ (Laplace Transform of $f''(t)$):

$$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)dt$$

Use integration by parts:

$$\begin{aligned} u &= e^{-st} & dv &= f''(t) \\ du &= -se^{-st} & v &= f'(t) \end{aligned}$$

Thus we have

$$\begin{aligned}
\mathcal{L}\{f''(t)\} &= \int_0^\infty e^{-st} f''(t)dt \\
&= e^{-st} f'(t)|_0^\infty + s\int_0^\infty e^{-st} f'(t)dt \\
&= \left[\lim_{b\to\infty} e^{-sb} f'(b)\right] - e^0 f'(0) + s\int_0^\infty e^{-st} f'(t)dt \\
&= 0 - f'(0) + s\mathcal{L}\{f'(t)\} \\
&= f'(0) + s\left[s\mathcal{L}\{f(t)\} - f(0)\right] \\
&= s^s\mathcal{L}\{f(t)\} - sf(0) - f'(0)
\end{aligned}$$

Example 187:

$$y'' - y = 0, y(0) = -1, y'(0) = 3$$

Using the substitution $Y(s) = \mathcal{L}\{y\}$,

$$\begin{aligned}
\mathcal{L}\{y'' - y\} &= \mathcal{L}\{0\} \\
\mathcal{L}\{y''\} - \dagger &= \mathcal{L}\{0\} \\
s^2Y(s) - sy(0) - y'(0) - Y(s) &= 0 \\
s^2Y(s) + s - 3 - Y(s) &= 0 \\
Y(s) &= \frac{3-s}{s^2-1}
\end{aligned}$$

Now we need to invert the Laplace transform. First, use partial fraction decomposition on the fraction to get

$$Y(s) = \frac{-2}{s+1} + \frac{1}{s-1}$$

So

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{\frac{-2}{s+1}\} + \mathcal{L}^{-1}\{\frac{1}{s-1}\} \\
&= -2\mathcal{L}^{-1}\{\frac{1}{s+1}\} + \mathcal{L}^{-1}\{\frac{1}{s-1}\} \\
&= -2e^{-t} + e^t
\end{aligned}$$

Example 188:

$$\begin{aligned}
f(t) &= [1 - H(t-2)]e^{-t} \\
&= e^{-t} - e^{-t}H(t-2) \\
&= \left\{ \begin{array}{ccc} e^{-t} & , & t < 2 \\ 0 & , & t > 2 \end{array} \right\}
\end{aligned}$$

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \mathcal{L}\{e^{-t}\} - \mathcal{L}\{e^{-t}H(t-2)\}\\ &= \mathcal{L}\{e^{-t}\} - e^{-2s}\mathcal{L}\{e^{-(t+2)}\}\\ &= \mathcal{L}\{e^{-t}\} - e^{-2s}\mathcal{L}\{e^{-t}e^{-2}\}\\ &= \mathcal{L}\{e^{-t}\} - e^{-2s-2}\mathcal{L}\{e^{-t}\}\\ &= \left[1-e^{-2s-2}\right]\mathcal{L}\{e^{-t}\}\\ &= \frac{1-e^{-2s-2}}{s+1}\end{aligned}$$

Definition 159: Let f and g be functions on the real line. Then the convolution of f and g, denoted $f * g$, is defined by

$$(f*g)(t) := \int_0^t f(\tau)g(t-\tau)d\tau$$

Platform $9\frac{3}{4}$ (Riemann-Lionville Fractional Integral of Order $\nu > 0$):

$$D^{\nu}y(t) := \frac{1}{\Gamma(\nu)}\int_0^t \frac{y(s)}{(t-s)^{\nu-1}}ds$$

Observation: Let's find $t * e^{-t}$:

$$\begin{aligned}t*e^{-t} &= \int_0^t \tau e^{-(t-\tau)}d\tau\\ &= \int_0^t \tau e^{\tau-t}dt\\ &= e^{-t}\int_0^t \tau e^{\tau}d\tau\end{aligned}$$

Use integration by parts:

$$\begin{aligned}u &= \tau \quad & dv &= e^{\tau}\\ du &= 1 \quad & v &= e^{\tau}\end{aligned}$$

So we have

$$\begin{aligned}t*e^{-t} &= e^{-t}\left[\tau e^{\tau}|_0^t - \int_0^t e^{\tau}d\tau\right]\\ &= e^{-t}\left[te^t - (e^t-1)\right]\\ &= t-1+e^{-t}\end{aligned}$$

Now let's take the Laplace transform of this, given that $\mathcal{L}\{t\} = \frac{1}{s^2}$, $\mathcal{L}\{e^{-t}\} = \frac{1}{1+s}$, and $\mathcal{L}\{1\} = \frac{1}{s}$.

$$\begin{aligned}\mathcal{L}\{t * e^{-t}\} &= \mathcal{L}\{t - 1 + e^{-t}\} \\ &= \mathcal{L}\{t\} - \mathcal{L}\{1\} + \mathcal{L}\{e^{-t}\} \\ &= \frac{1}{s^2} - \frac{1}{s} + \frac{1}{1+s} \\ &= \frac{1}{s^2(1+s)} \\ &= \mathcal{L}\{t\}\mathcal{L}\{e^{-t}\}\end{aligned}$$

Theorem 190 (Convolution Theorem):

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$$

supposing that everything exists.

Example 191:

$$b(t) = f(t)e^{-\mu t} + \int_0^t f(x)b(t-x)e^{-\mu x}dx$$

Editor's Note: This is the first instance of $f(x)$ since Example 124.

Letting $k(t) = f(t)e^{-\mu t}$,

$$b(t) = f(t)e^{-\mu t} + (b * k)(t)$$

So, letting $\mathcal{L}\{b(t)\} = B(s)$ and $\mathcal{L}\{k(t)\} =$

$$\begin{aligned}\mathcal{L}\{b(t)\} &= \mathcal{L}\{K\} + \mathcal{L}\{(b * k)(t)\} \\ B(s) &= K(s) + B(s)K(s) \\ B(s) &= \frac{K(s)}{1 - K(s)}\end{aligned}$$

So the solution is

$$b(s) = \mathcal{L}^{-1}\left\{\frac{K(s)}{1 - K(s)}\right\}$$

Editor's Note: Integrals with variable upper limits are called Volterra type integrals, while integrals with constant limits are called something else that I can't remember.

Platform $9\frac{3}{4}$ (Sturm-Lionville Problems):

...

Definition 192: A **Sturm-Lionville Problem** has the form:

$$\begin{aligned}(p(t)x')' + q(t)x + \lambda r(t)x &= 0 \\ \alpha x(a) - \beta x'(a) &= 0 \\ \gamma x(b) - \delta x'(b) &= 0\end{aligned}$$

$\alpha^2 + \beta^2 > 0$, $\gamma^2 + \delta^2 > 0$ And if you define

$$Lx := (p(t)x')' + q(t)x$$

$L : C^2(\mathbb{R}) \to C^2(\mathbb{R})$ then this becomes

$$\begin{aligned} Lx &= -\lambda r(t)x \\ \alpha x(a) - \beta x'(a) &= 0 \\ \gamma x(b) - \delta x'(b) &= 0 \end{aligned}$$

And the boundary conditions have special names that I didn't write down.

Definition 195: Assume that $r : [a, b] \to \mathbb{R}$ is continuous, and $r(t) \geq 0$ but not identically zero on $[a, b]$. The **inner product with respect to the weight function** r of the continuous functions x and y on $[a, b]$ is

$$< x, y >_r = \int_a^b x(t)y(t)r(t)dt$$

Definition 196: If $< x, y >_r = 0$, then x and y are orthogonal with respect to the weight function $r(t)$ on $[a, b]$.

Example 197: Let

$$\begin{aligned} x(t) &= t^2 \\ y(t) &= 4 - 5t \\ r(t) &= t \\ [a, b] &= [0, 1] \end{aligned}$$

$$\begin{aligned} < x, y >_r &= \int_0^1 t^2(4 - 5t)tdt \\ &= \int_0^1 4t^3 - 5t^4 dt \\ &= \left[t^4 - t^5\right]_0^1 \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

Therefore x and y are orthogonal.

Theorem 198: All eigenvalues of SLPs are real and simple. Corresponding to each eigenvalue there is a real-valued eigenfunction. Eigenfunctions corresponding to distinct eigenvaules of SLPs are orthogonal with respect to the weight function $r(t)$ on $[a, b]$.

Lemma 199 (Lagrange Identity): Assume x and y are sufficiently differentiable. Then

$$\begin{aligned} y(t)Lx(t) - x(t)Ly(t) &= [p(t)W[y(t), x(t)]]' \\ &=: \{y(t), x(t)\}' \end{aligned}$$

Proof:

$$\begin{aligned} [p(t)W[y(t), x(t)]]' &= [p(t)[y(t)x'(t) - x(t)y'(t)]]' \\ &= [p(t)y(t)x'(t) - p(t)x(t)y'(t)]' \\ &= y(t)\left(p(t)x'(t)\right)' + y'(t)p(t)x'(t) - x(t)\left(p(t)y'(t)\right)' - x'(t)p(t)y'(t) \\ &= y(t)\left[(p(t)x'(t))' + q(t)x(t)\right] - x(t)\left[(p(t)y'(t))' + q(t)y(t)\right] \\ &= yLx - xLy \end{aligned}$$

Now assume that (λ_1, x_1) and (λ_2, x_2) are eigenpairs for a SLP. By Lemma 199:

$$x_1 Lx_2 - x_2 Lx_1 = [p(t)W[x_1, x_2]]'$$

for $t \in [a, b]$. So,

$$(\lambda_1 - \lambda_2)r(t)x_1(t)x_2(t) = [p(t)W[x_1, x_2]]'$$

for $t \in [a, b]$. By FTC, we get that

$$\begin{aligned} (\lambda_1 - \lambda_2) < x_1, x_2 >_r &= \{p(t)W[x_1, x_2]\}_a^b \qquad = p(b)W[x_1, x_2](b) - p(a)W[x_1, x_2](a) \\ &= 0 \end{aligned}$$

Now to show that this last step is true, apply the initial conditions after expanding the Wronskians to get

$$\begin{aligned} p(a)W[x_1, x_2](a) &= p(a)[x_1(a)x_2'(a) - x_2(a)x_1'(a)] \\ &= p(a)[x_1(a)\frac{\alpha}{\beta}x_2(a) - x_2(a)\frac{\alpha}{\beta}x_1(a)] \\ &= 0 \end{aligned}$$

given $\beta \neq 0$. If $\beta = 0$ then actually this is trivially true. The point is it's true. Therefore provided $\lambda_1 \neq \lambda_2$, $x_1(t)$ and $x_2(t)$ are orthogonal with respect to $r(t)$ on $[a, b]$.
Now assume that (λ_0, x_0) is an eigenpair for the SLP. It turns out that $(\bar{\lambda}_0, \bar{x}_0)$ is also an eigen pair. So, It follows from above that

$$(\lambda_0 - \bar{\lambda}_0) < x_0, \bar{x}_0 >_r = 0$$

Thus, $\lambda_0 = \bar{\lambda}_0 \Leftarrow \lambda_0 \in \mathbb{R}$, because $< x_0, \bar{x}_0 >_r$ is not going to be zero unless $x_0 \equiv 0$.

Next, assume that λ_0 is an eigenvalue for the SLP and that x_1, x_2 are associated eigenfuntions. Note that as x_1, x_2 satisfy the boundary conditions it follows that

$$W[x_1, x_2](a) = 0$$

We also know that x_1, x_2 satisfy the same DE, namely $Lx = \lambda_0 r(t)x$. But then x_1, x_2 are LD on $[a, b]$. Therefore λ_0 is simple.

Theorem 200: If $q(t) \leq 0$ on $[a, b]$, $\alpha\beta \geq 0$, and $\gamma\delta \geq 0$, then al;l eigenvalues of the SLP are nonnegative.

Proof: Assume that λ_0 is an eigenvalue of the SLP. By Theorem 198, $\exists$ a real-valued eigenfunction x_0 associated to λ_0. Then

$$(p(t)x_0'(t))' + (\lambda_0 r(t) + q(t))x_0(t) = 0$$

for $t \in [a, b]$. Multiplying through by $x_0(t)$, we get

$$x_0(t)(p(t)x_0'(t))' + (\lambda_0 r(t) + q(t)){x_0}^2(t) = 0$$

$$\int_a^b x_0(px_0')'dt + \int_a^b \lambda_0 r x_0 dt + \int_a^b q{x_0}^2 dt = 0$$

$$\lambda_0 \int_a^b r x_0 dt = -\int_a^b x_0(px_0')'dt - \int_a^b q{x_0}^2 dt$$

$$\geq -\int_a^b x_0(px_0')'dt$$

Example 208:

$$x' = y$$
$$y' = -y + x - x^3$$

Note that $(0, 0)$ is an equilibrium point. For $x << 1$, $x^3 << 1$. So, we will analyze the linearized system

$$x' = y$$
$$y' = -y + x$$

which we can recast as

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

Looking at the determinant

$$\begin{vmatrix} 0 - \lambda & 1 \\ 1 & -1 - \lambda \end{vmatrix} = 0$$

$$\lambda + \lambda^2 - 1 = 0$$

So our eigenvalues are $-\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. Therfore this equilibrium is a saddle point. Another equilibrium is $(1, 0)$. Put

$$u := x - 1$$
$$v := y$$

Then

$$u' = x'$$
$$v' = y'$$

Then

$$u' = v$$
$$v' = -2u - v - 3u^2 - u^3$$

which is approximately equivalent to

$$u' = v$$
$$v' = -2u - v$$

close to the equilibrium. Our eigenvalues here are $-\frac{1}{2} \pm \frac{i\sqrt{7}}{2}$, so we have a stable spiral. The final equilibrium at $(-1, 0)$ is a stable spiral by similar analysis.

Platform 9$\frac{3}{4}$ (Formal Linearization):

$$x' = f(x, y)$$
$$y' = g(x, y)$$

Suppose that $\vec{x}_e$ is an equilibrium point for the system. Let's put $(u, v) := (x - x_e, y - y_e)$. Then

$$u' = f(u + x_e, v + y_e)$$
$$v' = g(u + x_e, v + y_e)$$

Assuming that f and g are nice, we get the following first-order Taylor polynomials:

$$f(x, y) = f(\vec{x}_e) + (x - x_e)f_x(\vec{x}_e) + (y - y_e)f_y(\vec{x}_e) + R_1(x, y)$$

and

$$g(x, y) = f(\vec{x}_e) + (x - x_e)g_x(\vec{x}_e) + (y - y_e)g_y(\vec{x}_e) + R_1(x, y)$$

It must be the caese (c.f. Colley, Section 4.1) that

$$\lim_{\vec{x}\to\vec{x}_e} \frac{R_1(\vec{x})}{||\vec{x}||}, \frac{R_2(\vec{x})}{||\vec{x}||} = 0$$

Thus

$$\begin{aligned} u' &= uf_x(\vec{x}_e) + vf_y(\vec{x}_e) \\ v' &= ug_x(\vec{x}_e) + vg_y(\vec{x}_e) \end{aligned}$$

which is actually

$$\begin{aligned} \vec{u}' &= J(\vec{x}_e)\vec{u} \\ &= D\vec{f}(\vec{x}_e)\vec{u} \end{aligned}$$

Example 209:

$$\begin{aligned} x' &= y \\ y' &= x(x-4) \end{aligned}$$

The only equilibrium points are $\vec{x}_e^1 = (0,0)$ and $\vec{x}_e^2 = (4,0)$. Note that the Jacobian here is

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ 2x-4 & 0 \end{bmatrix}$$

and therefore

$$J(\vec{x}_e^i) = \begin{bmatrix} 0 & 1 \\ 2x_e^i-4 & 0 \end{bmatrix}$$

So

$$J(\vec{x}_e^1) = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2}^1 = \pm 2i$$

and

$$J(\vec{x}_e^2) = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2}^2 = \pm 2$$

For $\vec{x}_e^1$, it is guaranteed that the equilibrium is a neutrally stable center. For $\vec{x}_e^2$, it is uncertain that we will get an unstable saddle but we do, as shown by a picture.

Example 210:

$$\begin{aligned} x' &= y - x||\vec{x}|| = y - x\sqrt{x^2+y^2} \\ y' &= -x - y||\vec{x}|| = -x - y\sqrt{x^2-y^2} \end{aligned}$$

There is an equilibrium at $(0,0)$.

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

So our eigenvalues are $\lambda_{1,2} = \pm i$. Since these are pure imaginary, we can't tell if it will spiral in or out. Think about it and split the vector field into two separate vector fields across the addition.

Platform $9\frac{3}{4}$ (Floquet Theory):

Theorem 211 (Jordan Canonical Form): If A is a constant $n \times n$ real matrix with eigenvalues $\lambda_1, ..., \lambda_n$, then there is a nonsingular constant matrix $P \ni A = PJP^{-1}$, where J has the block diagonal form

$$J := \begin{bmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_k \end{bmatrix}$$

and J_i is either the 1×1 matrix $[\lambda_i]$ or

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}, 1 \leq i \leq k$$

Theorem 212 (Logarithm of a Matrix): If C is an $n \times n$ nonsingular matrix, then there exaists a matrix B such that

$$e^B = C$$

Theorem 213 (Floquet's Theorem): If Φ is a fundamental matrix for the Floquet system

$$\vec{x}' = A(t)\vec{x}$$

where A is continuous on $\mathbb{R}$ and ω-periodic, then

$$\Psi(t) := \Phi(t+\omega)$$

is also a fundamental matrix, $\forall t \in \mathbb{R}$. Further, there is a nonsingular, continuously differentiable matrix function $P(t)$, where P is ω-periodic, and an $n \times n$ constant matrix B (possibly complex) such that

$$\Psi(t) = P(t)e^{Bt}, \forall t \in \mathbb{R}$$

Proof: Assume that Φ is a fundamental matrix for the Floquet system. Put

$$\Psi(t) := \Psi(t+\omega)$$

Note that:

1. $\det \Psi(t) = \det \Phi(t+\omega) \neq 0, \forall t \in \mathbb{R}$

2. $\Psi'(t) = \Phi'(t+\omega) = A(t+\omega)\Phi(t+\omega) = A(t)\Psi(t)$

whence $\Psi(t)$ is also a fundamental matrix. Moreover, since Φ and Ψ are fundamental matrices for the same system, it can be shown that there exists a nonsingular constant $n \times n$ matrix $C \ni \Psi(t) = \Phi(t+\omega) = \Phi(t)C, \forall t \in \mathbb{R}$. By Theorem 212, $\exists B \ni e^{B\omega} = C$. Put

$$P(t) := \Phi(t)e^{-Bt}$$

Clearly, $P \in C^1(\mathbb{R}, \mathbb{R}^{n\times n})$. Also:

$$\begin{aligned} P(t+\omega) = \Phi(t+\omega)e^{-Bt-B\omega} &= \Phi(t)Ce^{-B\omega}e^{-Bt} \\ &= \Phi(t)e^{-Bt} \\ &= P(t) \end{aligned}$$

$\forall t \in \mathbb{R}$ and

$$P(t)e^{Bt} = \Phi(t)$$

Definition 214: Let Φ be a fundamental matrix for a Floquet system $\vec{x}' = A(t)\vec{x}$. Then the eigenvalues of $C : \Phi^{-1}(0)\Phi(\omega)$ are called the **Floquet multipliers** for the Floquet system.

Proposition 215: Floquet multipliers are well-defined.

Proof: Let Φ and Ψ be two different fundamental amatrices for the Floquet system. Put

$$\begin{aligned} C &:= \Phi^{-1}(0)\Phi(\omega) \\ D &:= \Psi^{-1}(0)\Psi(\omega) \end{aligned}$$

Firstly, there exists a constant, nonsingular $n \times n$ matrix such that

$$\Psi(t) = \Phi(t)M, \forall t \in \mathbb{R}$$

But then

$$\begin{aligned} D &= \Psi^{-1}(0)\Psi(\omega) \\ &= [\Phi(0)M]^{-1}[\Phi(\omega)M] \\ &= M^{-1}\Phi^{-1}(0)\Phi(\omega)M \\ &= M^{-1}CM \end{aligned}$$

So C is similar to D. Since similar matrices have the same eigenvalues, C and D have the same eigenvalues and Floquet multipliers are well-defined.

Example 216:

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 0 & \frac{\cos t+\sin t}{2+\sin t-\cos t} \end{bmatrix} \vec{x}$$

$$\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= \left(\frac{\cos t+\sin t}{2+\sin t-\cos t}\right) x_2 \end{aligned}$$

We can solve this by seperation of variables.

$$\begin{aligned} \int \frac{1}{x_1} &= \int \frac{\cos t+\sin t}{2+\sin t-\cos t} dt \\ \ln|x_2| &= \ln|2+\sin t-\cos t| \\ x_2 &= \beta(2+\sin t-\cos t) \end{aligned}$$

By IFM,

$$x_1 = \alpha e^t - \beta(2+\sin t)$$

So our fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -2-\sin t & e^t \\ 2+\sin t-\cos t & 0 \end{bmatrix}$$

So

$$\Phi^{-1} = \frac{1}{-e^t(2+\sin t-\cos t)} \begin{bmatrix} 0 & -e^t \\ -2-\sin t+\cos t & -2-\sin t \end{bmatrix}$$

and at $t = 0$, it is

$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

Note that $\omega = 2\pi$, so

$$\Phi(\omega) = \Phi(2\pi) = \begin{bmatrix} -2 & e^{2\pi} \\ 1 & 0 \end{bmatrix}$$

So

$$\begin{aligned} \Phi^{-1}(0)\Phi(2\pi) &= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & e^{2\pi} \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi} \end{bmatrix} \end{aligned}$$

So our Floquet multipliers are 1 and $e^{2\pi}$.

Theorem 217: Let $\Phi(t) := P(t)e^{Bt}$ be as in Floquet's Theorem. Then $\vec{x}$ is a solution of the Floquet system if and only if the vector $\vec{y}$ defined by

$$\vec{y}(t) := P^{-1}(t)\vec{x}(t), t \in \mathbb{R}$$

solves $\vec{y}' = B\vec{y}$.

Proof: Assume that $\vec{x}$ solves the Floquet system. Then $\vec{x}(t) = \Phi(t)\vec{x}_0$, where $\vec{x}_0 \in \mathbb{R}^n$ is fixed. Put

$$\vec{y}(t) := P^{-1}(t)\vec{x}(t)$$

Thus,

$$\begin{aligned} \vec{y}(t) &= P^{-1}(t)\vec{x}(t) \\ &= P^{-1}(t)\Phi(t)\vec{x}_0 \\ &= e^{Bt}\vec{x}_0 \end{aligned}$$

which proves the claim in the forward direction. Conversely, assume that $\vec{y}$ solves $\vec{y}' = B\vec{y}$. Put

$$\vec{x}(t) := P(t)\vec{y}(t)$$

Also, since $\vec{y}$ solves $\vec{y}' = B\vec{y}$, $\vec{y} = e^{Bt}\vec{y}_0$ for some fixed $\vec{y}_0 \in \mathbb{R}^n$. But then

$$\begin{aligned} \vec{x}(t) &= P(t)\vec{y}(t) \\ &= P(t)e^{Bt}\vec{y}_0 \\ &= \Phi(t)\vec{y}_0 \end{aligned}$$

which proves the claim.

Theorem 218: Let $\mu_1, ..., \mu_n$ be the Floquet multipliers for the Floquet system $\vec{x}' = A(t)\vec{x}$. Then the trivial solution is:

1. asymptotically stable on $[0, \infty)$ if and only if $|\mu_i| < 1, \forall i$
2. atable on $[0, \infty)$ provided that $|\mu_i| \leq 1, \forall i$, and whenever $|\mu_i| = 1$, μ_i is a simple eigenvalue
3. unstable on $[0, \infty)$ provided that there is an index i_0, $1 \leq i_0 \leq n$, such that $|\mu_i| > 1$.

Theorem 219: Assume $\mu_1, ..., \mu_n$ are the Floquet multipliers for the Floquet system $\vec{x}' = A(t)\vec{x}$. Then:

$$\prod_{i=1}^{n} \mu_i = e^{\int_0^\infty \mathrm{tr}|A(t)|dt}$$

Theorem 220: The number μ_0 is a Floquet multiplier of the Floquet system for the Floquet system $\vec{x}' = A(t)\vec{x}$ if and only if there exists a nontrivial solution $\vec{x}$ such that $\vec{x}(t+\omega) = \mu_0\vec{x}(t), \forall t \in \mathbb{R}$. Consequently, the Floquet system admits a periodic solution of period ω if and only if $\mu_0 = 1$ is a Floquet multiplier.

Example 221 (Hill's Equation): Consider the scalar differential equation

$$y'' + q(t)y = 0$$

where q is continuous and ω-periodic on $\mathbb{R}$.

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -q(t) & 0 \end{bmatrix} \vec{x}$$

$$\mu_1 \mu_2 = \prod_{i=1}^{2} \mu_i = e^{\int_0^\infty 0dt} = e^0 = 1$$

Example 222 (Mathieu's Equation):

$$y'' + (\alpha + \beta \cos t)y = 0$$

We get from Example 221 that

$$\mu_1(\alpha, \beta)\mu_2(\alpha, \beta) = 1$$

Let $\gamma(\alpha, \beta) := \mu_1(\alpha, \beta) + \mu_2(\alpha, \beta)$. Then

$$\mu^2 - \gamma(\alpha, \beta)\mu + 1 = 0$$

So

$$\mu_{1,2} = \frac{\gamma \pm \sqrt{\gamma^2 - 4}}{2}$$

And you can analyze this case by case.

Example 223:

1. Show that if $\mu_0 = -1$ is a multiplier if the Floquet system $\vec{x}' = A(t)\vec{x}$, then there is a nontrivial solution of period 2ω.

2. Show that

$$\vec{x}(t) = \begin{bmatrix} -e^{t/2} & \cos t \\ e^{t/2} & \sin t \end{bmatrix}$$

is a solution of the Floquet system

$$\vec{x}' = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\cos t \sin t \\ -1 - \frac{3}{2}\sin t \cos t & -1 + \frac{3}{2}\sin^2 t \end{bmatrix}$$

Find a Floquet multiplier for this system, find the other one, and analyze the stability of $\vec{x} = \vec{0}$. Then show that for each t, the coefficient matrix has eigenvalues with a negative real part.

3. Show that if μ_0 is a Floquet multiplier for the Floquet system

$$\vec{x}' = A(t)\vec{x}$$

then there is a umber p_0 (called a Floquet exponent) such that there is a nontrivial solution $\vec{x}_0(t)$ of the original system such that

$$\vec{x}_0(t) = e^{p_0 t} p_0(t)$$

where $p_0 \in C^1(\mathbb{R}, \mathbb{R}^{\not{\vDash}})$ and is ω-periodic.

Proof of Theorem 220: For the forward direction, assume that μ_0 is a Floquet multiplier for the Floquet system. Then μ_0 is an eigenvalue of the matrix $C := \Phi^{-1}(0)\Phi(\omega)$, where Φ is a fundamental matrix for the Floquet system. Let $\vec{x}_0$ be the eigenvector associated to μ_0 and put

$$\vec{x}(t) := \Phi(t)\vec{x}_0, \forall t \in \mathbb{R}$$

It follows that $\vec{x}$ is nontrivial, $\vec{x}$ solves $\vec{x}' = A(t)\vec{x}$, and

$$\begin{aligned} \vec{x}(t+\omega) &= \Phi(t+\omega)\vec{x}_0 \\ &= \Phi(t)C\vec{x}_0 \\ &= \Phi(t)\mu_0\vec{x}_0 \\ &= \mu_0\vec{x}(t), \forall t \in \mathbb{R} \end{aligned}$$

as desired. Conversely, assume that $\exists$ a nontrivial solution $\vec{x}(t)$ such that

$$\vec{x}(t+\omega) = \mu_0\vec{x}(t), \forall t \in \mathbb{R}$$

Let $\Psi(t)$ be a fundamental matrix for the Floquet system. Then $\vec{x} = \Psi(t)\vec{y}_0, \forall t \in \mathbb{R}, \vec{y}_0 \in \mathbb{R}^n \backslash \{\vec{0}\}$. By Floquet's Theorem, $\Psi(t+\omega)$ is also a fundamental matrix for each $t \in \mathbb{R}$. So, it follows that

$$\begin{aligned} \vec{x}(t+\omega) &= \mu_0\vec{x}(t) \\ \Psi(t+\omega)\vec{y}_0 &= \mu_0\Psi(t)\vec{y}_0 \\ \Psi(t)D\vec{y}_0 &= \Psi(t)\mu_0\vec{y}_0 \end{aligned}$$

whence, by definition, is a Floquet multiplier, as claimed.

Theorem 224: If Φ is a fundamental matrix for $\vec{x} = A\vec{x}$, then $\Psi = \Phi C$, where C is an arbitrary $n \times n$ constant nonsingular matrix, is a general fundamental matrix of $\vec{x}' = A\vec{x}$.

Proof: Assume that Φ and Ψ are as given in the statement of the theorem. Then

1. $\det\Psi = \det[\Phi C] = \det\Phi \det C, \forall t \in \mathbb{R}$

2. $\Psi'(t) = \Phi'(t)C = A(t)\Phi(t)C = A(t)\Psi(t), \forall t \in \mathbb{R}$

whence $\Psi(t)$ is also a fundamental matrix. Now to show that every fundamental matrix has this general form, assume that Ψ is an arbitrary but fixed fundamental matrix of $\vec{x}' = A(t)\vec{x}$. Let $t_0 \in I$ and let

$$C_0 := \Phi^{-1}(t_0)\Psi(t_0)$$

So, C_0 is nonsingular and $\Psi(t_0) = \Phi(t_0)C_0$. Also, $\Psi'(t) = A(t)\Psi(t)$. By uniqueness of solutions to IVPs, $\Psi(t) \equiv \Phi(t)C$.